Summary of What’s Known about Convergence in Concave Games

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1 Concave Games

**Definition 1** (Narrow Definition for single dimensionaly strategy and orthogonal strategy space). A game is called concave if each player $i$ chooses a real quantity $x_i \in [a_i, b_i]$ to maximize utility is $u_i(\vec{x})$, where $u_i(\vec{x})$ is concave in $x_i$.

**Theorem 1.1** (Rosen). Concave games have (possibly multiple) Nash Equilibrium.

2 Socially Concave Games

**Definition 2** (Even-Dar et. al.). A concave game is socially concave if each utility $u_i(x_i, x_{-i})$ is convex in $x_{-i}$, and there exists $\lambda_i > 0$ such that $g(\vec{x}) = \sum_i \lambda_i u_i(\vec{x})$ is concave.

**Theorem 2.1** (Even-Dar et. al.). If every player in a socially concave game plays according to a procedure with external regret bound $R_i(t)$, then at time $t$,

1. The average strategy vector $\hat{x}_t$ is an $\epsilon_t$-Nash equilibrium, where $\epsilon_t$ depends on the regret bound and minimum $\lambda_i$.

2. The average utility of each player $i$ is $\epsilon_t$ close to her utility at $\hat{x}_t$ (the average vector of strategies).

3 Rosen’s Framework

**Definition 3** (Rosen). In a concave game, players follow the continuous best reply (BR) dynamic if there exists constants $\lambda_i > 0$ s.t. each player $i$ adjusts $x_i$ continuously with $\frac{dx_i}{dt} = \lambda_i \frac{\partial u_i(\vec{x})}{\partial x_i}$.

**Definition 4** (Zinkevich). In a concave game, players follow GIGA if there exists constants $\lambda_i > 0$ s.t. each player $i$ adjusts $x_i$ every discrete time step $t$ with $\Delta x_i = \frac{\lambda_i}{\sqrt{t}} \frac{\partial u_i(\vec{x})}{\partial x_i}$.

**Theorem 3.1** (Rosen). Define $n \times n$ matrix function $G$ in which $G_{ij} = \lambda_i \frac{\partial^2 u_i(\vec{x})}{\partial x_i \partial x_j}$, for some constant choices of $\lambda_i > 0$. Then if $G + G^T$ is strictly negative definite, then the Nash equilibrium is unique, and starting at any $\vec{x}_0$, the continuous BR dynamic converges to the unique equilibrium.
4 Incremental Results

Theorem 4.1 (Moulin). If a concave game satisfies the dominance solvability condition
\[ -\frac{\partial^2 u_i}{\partial x_i} \geq \sum_j |\frac{\partial^2 u_i}{\partial x_i \partial x_j}| \]
then it satisfies Rosen’s conditions in Theorem 3.1.

Definition 5. A socially concave game is strict socially concave if any one of the following is satisfied:
1. For all \( i \) \( u_i(x_i, x_{-i}) \) is strictly concave in \( x_i \).
2. There exists \( i, j \) such that \( u_i(x_i, x_{-i}) \) is strictly convex in \( x_{-i} \) and \( u_j(x_j, x_{-j}) \) is strictly convex in \( x_{-j} \).
3. \( f(x) = \sum_i \lambda_i u_i(x) \) is strictly concave in \( x \).

Theorem 4.2. Strict socially concave games satisfy Rosen’s condition in Theorem 3.1.

Corollary 4.3. In strict socially concave games, the continuous BR dynamic converge to unique equilibrium.

Theorem 4.4. If the conditions in 3.1 hold, and all the utility functions’ first, second and third derivatives are bounded, then GIGA converges to unique equilibrium.

Corollary 4.5. In strict socially concave games, when players follow GIGA, not only do the statements in Theorem 2.1 hold, but the strategies converge to unique Nash equilibrium.

5 Negative Results

Theorem 5.1. There exists a 2 player non-strict socially concave game in which the continuous BR dynamic does not converge.

Theorem 5.2. Continuous better reply dynamic, in which players move in the direction of \( \frac{\partial u_i(x)}{\partial x_i} \) but not necessarily with fixed proportional speed, need not converge even under Rosen’s conditions in Theorem 3.1.

Theorem 5.3. In 2 player socially concave games, sequential best reply (player best reply in turns) need not converge.

6 Open Questions

- What happens to Rosen’s condition when negative definite is replaced with negative semi-definite? (This would include all socially concave games.) Can we make the statements in Theorem 2.1? (Does the strategies GIGA under average to \( \epsilon \)-Nash equilibrium? What about the average utilities?)