Inequalities?

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- Many tools available (see formula sheet)

Fundamental problem solving ideas:
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Fundamental problem solving ideas:
- Smoothing
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Fundamental problem solving ideas:

1. Smoothing
2. Substitution
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Fundamental problem solving ideas:

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2. Substitution
3. Clever manipulation
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Fundamental problem solving ideas:

1. Smoothing
2. Substitution
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4. Bash (not recommended)
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Fundamental problem solving ideas:

1. Smoothing
2. Substitution
3. Clever manipulation
4. Bash (not recommended)

Due to the nature of the topic, there will be quite a mess of expressions, so don’t feel that you have to follow every line.
Basic tools

Theorem
\( x^2 \geq 0 \) with equality iff \( x = 0 \).
Basic tools

**Theorem**

\[ x^2 \geq 0 \text{ with equality iff } x = 0. \]

**Example**

\[ x^2 + y^2 + z^2 \geq xy + yz + xz \text{ because } (x - y)^2 + (y - z)^2 + (z - x)^2 \geq 0. \]
Basic tools

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**Theorem**

**AM-GM** If \( x_1, \ldots, x_n \) are positive real numbers, then

\[
\frac{x_1 + \cdots + x_n}{n} \geq \sqrt[n]{x_1 \cdots x_n}
\]

with equality iff \( x_1 = x_2 = \cdots = x_n \).
**Basic tools**

**Theorem**

$x^2 \geq 0$ with equality iff $x = 0$.

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$x^2 + y^2 + z^2 \geq xy + yz + xz$ because $(x - y)^2 + (y - z)^2 + (z - x)^2 \geq 0$.

**Theorem**

AM-GM If $x_1, \cdots, x_n$ are positive real numbers, then

$$\frac{x_1 + \cdots + x_n}{n} \geq \sqrt[n]{x_1 \cdots x_n}$$

with equality iff $x_1 = x_2 = \cdots = x_n$.

**Example**

$x^3 + y^3 + z^3 \geq 3xyz$
Basic tools

**Theorem**
(Cauchy-Schwarz) For any real numbers $a_1, \cdots, a_n, b_1, \cdots, b_n$,

$$(a_1^2 + \cdots + a_n^2)(b_1^2 + \cdots + b_n^2) \geq (a_1 b_1 + \cdots + a_n b_n)^2$$

with equality iff the two sequences are proportional. ($\|\vec{a}\| \|\vec{b}\| \geq \vec{a} \cdot \vec{b}$.)
Basic tools

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Example

$$\frac{1}{a} + \frac{1}{b} + \frac{4}{c} + \frac{16}{d} \geq \frac{64}{a + b + c + d}$$

because

$$(a + b + c + d)\left(\frac{1}{a} + \frac{1}{b} + \frac{2^2}{c} + \frac{4^2}{d}\right) \geq (1 + 1 + 2 + 4)^2$$
Basic tools

Theorem
(Jensen) Let $f$ be a convex function. Then for any $x_1, \cdots, x_n \in I$ and any non-negative reals $w_1, \cdots, w_n$, $\sum_i w_i = 1$

$$w_1 f(x_1) + \cdots + w_n f(x_n) \geq f(w_1 x_1 + \cdots + w_n x_n)$$

If $f$ is concave, then the inequality is flipped.
Basic tools

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**Example**
One proof of the AM-GM inequality uses the fact that $f(x) = \log(x)$ is concave, so

$$\frac{1}{b} (\log x_1 + \cdots + \log x_n) \leq \log \frac{x_1 + \cdots + x_n}{n}$$

from which AM-GM follows by taking exponents of both sides.
Basic tools

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For other tools, see the formula sheet.
Idea 1: Smoothing

By altering terms and arguing what happens, we can sometimes reduce proving $A \geq B$ in general to checking a canonical case.
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For example, while fixing $B$, say that we can decrease $A$ by moving terms closer, then it suffices to check the case when all terms are equal.

Example
Prove the AM-GM inequality

\[
\left(\sum_i x_i^n\right)^{\frac{1}{n}} \geq \prod_i x_i
\]

Proof.
Let $\bar{x} = \frac{1}{n} \sum x_i$. Say that $x_i < \bar{x}$ and $x_j > \bar{x}$. Consider replacing $(x_i, x_j)$ by $(\bar{x}, 2\bar{x})$. Note that $x_i x_j \leq \bar{x}(2\bar{x} - x_i)$. Hence, this fixes LHS but increases RHS. So for fixed LHS, the RHS is maximized when $x_i = \bar{x} \forall i$. 
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\left( \frac{\sum_i x_i}{n} \right)^n \geq \prod_i x_i
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Connection to convexity

Usually smoothing is equivalent to arguing about a convex function: for fixed $\sum_i x_i$ and $f$ convex, $\sum_{i=1}^n f(x_i)$ is minimized when all $x_i$’s are equal.
Connection to convexity

Usually smoothing is equivalent to arguing about a convex function: for fixed $\sum_i x_i$ and $f$ convex, $\sum_i^n f(x_i)$ is minimized when all $x_i$’s are equal.

Sometimes the function is not convex, in which case the argument needs to be more intricate.
Another example

**Example**

If $a, b, c, d, e$ are real numbers such that

\[
\begin{align*}
    a + b + c + d + e &= 8 \quad (1) \\
    a^2 + b^2 + c^2 + d^2 + e^2 &= 16 \quad (2)
\end{align*}
\]

What is the largest possible value of $e$?
Another example

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What is the largest possible value of $e$?

Solution.
Relax the second constraint to $a^2 + b^2 + c^2 + d^2 + e^2 \leq 16$ (2*). Call a 5-tuple valid if it satisfies (1) and (2*). We seek the valid 5-tuple with the largest possible $e$. 
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**Solution.**

Relax the second constraint to \( a^2 + b^2 + c^2 + d^2 + e^2 \leq 16 \) (2*). Call a 5-tuple valid if it satisfies (1) and (2*). We seek the valid 5-tuple with the largest possible \( e \). For any valid \( (a, b, c, d, e) \), setting \( k = \frac{(a+b+c+d)}{4} \), \( (k, k, k, k, e) \) is also valid (smoothing). So we just need to find the largest \( e \) s.t. for some \( k \)

\[
\begin{align*}
    4k + e &= 8 \\
    4k^2 + e &\leq 16
\end{align*}
\]

\[
\implies \left( \frac{8 - e}{4} \right)^2 = k^2 \leq \frac{16 - e^2}{4}
\]

\[
\implies (5e - 16)e \leq 0 \quad \Rightarrow 0 \leq e \leq \frac{16}{5}
\]
Another example

Example
If \( a, b, c, d, e \) are real numbers such that

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\begin{align*}
    a + b + c + d + e &= 8 \quad \text{(1)} \\
    a^2 + b^2 + c^2 + d^2 + e^2 &= 16 \quad \text{(2)}
\end{align*}
\]

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Relax the second constraint to \( a^2 + b^2 + c^2 + d^2 + e^2 \leq 16 \) \((2^*)\). Call a 5-tuple valid if it satisfies (1) and \((2^*)\). We seek the valid 5-tuple with the largest possible \( e \). For any valid \((a, b, c, d, e)\), setting \( k = \left(\frac{a+b+c+d}{4}\right) \), \((k, k, k, k, e)\) is also valid (smoothing). So we just need to find the largest \( e \) s.t. for some \( k \)

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\]

\[
\Rightarrow (5e - 16)e \leq 0 \quad \Rightarrow 0 \leq e \leq \frac{16}{5}
\]

Conversely, \((\frac{6}{5}, \frac{6}{5}, \frac{6}{5}, \frac{6}{5}, \frac{16}{5})\) satisfies the original equation, so the largest possible value is \(\frac{16}{5}\). \(\square\)
Idea 2: Substitution

Use substitutions to transform the given inequality into a simpler or “nicer” form.
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Commonly used substitutions

- Simple manipulations
- Triangle related substitutions
- Homogenization
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Use substitutions to transform the given inequality into a simpler or “nicer” form.

Commonly used substitutions

- Simple manipulations
- Triangle related substitutions (cool)
- Homogenization (will show later)
Triangle related substitutions

- If $a, b, c$ are sides of a triangle, then let $x = (b + c - a)/2$, $y = (a + c - b)/2$, $z = (a + b - c)/2$, so that $a = y + z$, $b = x + z$, $c = x + y$, and $x, y, z$ are arbitrary positive real numbers.
Triangle related substitutions

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- If $a^2 + b^2 = 1$, let $a = \cos \theta$, $b = \sin \theta$. 
Triangle related substitutions

If \(a, b, c\) are sides of a triangle, then let  
\[ \begin{align*} 
    x &= \frac{(b + c - a)}{2}, \\
    y &= \frac{(a + c - b)}{2}, \\
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\end{align*} 
\]  
so that  
\[ a = y + z, \quad b = x + z, \quad c = x + y, \]  
and \(x, y, z\) are arbitrary positive real numbers.

If \(a^2 + b^2 = 1\), let  
\[ a = \cos \theta, \quad b = \sin \theta. \]

If \(a + b + c = abc\), \(a, b, c > 0\), let  
\[ a = \tan A, \quad b = \tan B, \quad c = \tan C, \]  
where \(A, B, C\) are angles in a triangle.
Triangle related substitutions

- If $a$, $b$, $c$ are sides of a triangle, then let $x = (b + c - a)/2$, $y = (a + c - b)/2$, $z = (a + b - c)/2$, so that $a = y + z$, $b = x + z$, $c = x + y$, and $x$, $y$, $z$ are arbitrary positive real numbers.

- If $a^2 + b^2 = 1$, let $a = \cos \theta$, $b = \sin \theta$.

- If $a + b + c = abc$, $a, b, c > 0$, let $a = \tan A$, $b = \tan B$, $c = \tan C$, where $A, B, C$ are angles in a triangle.

- If $a^2 + b^2 + c^2 + 2abc = 1$, let $a = \cos A$, $b = \cos B$, $c = \cos C$, where $A, B, C$ are angles in a triangle.
Slick example

Example
For positive real numbers $a, b, c$ with $a + b + c = abc$, show that

\[
\frac{1}{\sqrt{1 + a^2}} + \frac{1}{\sqrt{1 + b^2}} + \frac{1}{\sqrt{1 + c^2}} \leq \frac{3}{2}
\]

Proof.
WLOG, let $a = \tan A$, $b = \tan B$, $c = \tan C$, where $A, B, C$ are angles in a triangle. The inequality is equivalent to

\[
\cos A + \cos B + \cos C \leq \frac{3}{2}
\]

But $\cos A + \cos B + \cos C = 1 + 4 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}$, so inequality follows from $f(x) = \log \sin x$ being concave, where $0 \leq x \leq \frac{\pi}{2}$. \qed
Clever Manipulation

Manipulate the algebra in a possibly strange way and magically yield the result.
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Guidelines:
- Be creative; try random things
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Can only learn this through experience.
How did they come up with that?

Example

(IMO 2005 P3) Let $x, y, z$ be three positive reals such that $xyz \geq 1$. Prove that

$$\frac{x^5 - x^2}{x^5 + y^2 + z^2} + \frac{y^5 - y^2}{x^2 + y^5 + z^2} + \frac{z^5 - z^2}{x^2 + y^2 + z^5} \geq 0$$
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Solution.

By Cauchy-Schwarz

$$(x^5 + y^2 + z^2)\left(\frac{1}{x} + y^2 + z^2\right) \geq (x^2 + y^2 + z^2)^2$$

So
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$$\frac{\frac{1}{x} + y^2 + z^2}{x^2 + y^2 + z^2} \geq \frac{x^2 + y^2 + z^2}{x^5 + y^2 + z^2}$$
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So

$$\frac{1}{x} + y^2 + z^2 \geq \frac{x^2 + y^2 + z^2}{x^5 + y^2 + z^2}$$

$$\implies \frac{yz - x^2}{x^2 + y^2 + z^2} + 1 \geq \frac{x^2 - x^5}{x^5 + y^2 + z^2} + 1$$
How did they come up with that?

Example

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So

$$\frac{1}{x} + y^2 + z^2 \geq \frac{x^2 + y^2 + z^2}{x^5 + y^2 + z^2}$$

$$\Rightarrow \frac{yz - x^2}{x^2 + y^2 + z^2} + 1 \geq \frac{x^2 - x^5}{x^5 + y^2 + z^2} + 1$$

$$\Rightarrow LHS \geq \frac{x^2 + y^2 + z^2 - xy - yz - xz}{x^2 + y^2 + z^2} \geq 0$$
Idea 4: Bash (for symmetric inequalities)

Not really an idea, but what you do when you run out of ideas and feel like an algebraic workout.
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Not really an idea, but what you do when you run out of ideas and feel like an algebraic workout.

Preliminaries:

- Symmetric notation: \( i.e. \)

\[
(2, 1, 1) = \sum_{\text{sym}} x^2yz = x^2yz + x^2zy + y^2xz + y^2zx + z^2xy + z^2yx
\]
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Preliminaries:

- Symmetric notation: \( i.e. \)
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  (2, 1, 1) = \sum_{\text{sym}} x^2 yz = x^2 yz + x^2 zy + y^2 xz + y^2 zx + z^2 xy + z^2 yx
  \]

- Homogenization: \( i.e. \) If \( xyz = 1 \), then
  \[
  \frac{x^5 - x^2}{x^5 + y^2 + z^2} = \frac{x^5 - x^3 yz}{x^5 + xy^3 z + xyz^3}
  \]

  Equivalently one can substitute \( x = \frac{bc}{a}, y = \frac{ac}{b}, z = \frac{ab}{c} \).
**Definition**

(Majorization) Sequence $x_1, \cdots, x_n$ is said to *majorize* sequence $y_1, \cdots, y_n$ if

\[
\begin{align*}
x_1 & \geq y_1 \\
x_1 + x_2 & \geq y_1 + y_2 \\
\vdots & \quad \vdots \\
\sum_{i=1}^{n-1} x_i & \geq \sum_{i=1}^{n-1} y_i \\
\sum_{i=1}^{n} x_i & = \sum_{i=1}^{n} y_i
\end{align*}
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&\quad \quad \quad \quad \cdots \\
&\sum_{i=1}^{n-1} x_i \geq \sum_{i=1}^{n-1} y_i \\
&\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i
\end{align*}
$$

**Theorem**

(Muirhead) Suppose the sequence $a_1, \cdots, a_n$ majorizes the sequence $b_1, \cdots, b_n$. Then for any positive reals $x_1, \cdots, x_n$,

$$
\sum_{\text{sym}} x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n} \geq \sum_{\text{sym}} x_1^{b_1} x_2^{b_2} \cdots x_n^{b_n}
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where the sums are taken over all permutations of the $n$ variables.
Ultimate Bash Toolbox

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\]

where the sums are taken over all permutations of the $n$ variables.

Example
$\sum_{sym} x^4 y \geq \sum_{sym} x^3 yz$ (In simplified notation, $(4,1,0) \geq (3,1,1)$)
Theorem
(Schur) Let $a, b, c$ be nonnegative reals and $r > 0$. Then

$$a^r (a - b)(a - c) + b^r (b - c)(b - a) + c^r (c - a)(c - b) \geq 0$$

with equality iff $a = b = c$ or some two of $a, b, c$ are equal and the other is 0. ($\sum_{sym} a^{r+2} + \sum_{sym} a^r bc \geq 2 \sum_{sym} a^{r+1} b$)
Theorem
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a^r(a - b)(a - c) + b^r(b - c)(b - a) + c^r(c - a)(c - b) \geq 0
\]
with equality iff \(a = b = c\) or some two of \(a, b, c\) are equal and the other is 0. \((\sum_{\text{sym}} a^{r+2} + \sum_{\text{sym}} a^r bc \geq 2 \sum_{\text{sym}} a^{r+1} b)\)

Example
\[
\sum_{\text{sym}} x^3 + \sum_{\text{sym}} xyz \geq 2 \sum_{\text{sym}} x^2 y
\]
Equivalently
\[
2(x^3 + y^3 + z^3 + 3xyz) \geq 2(x^2(y + z) + y^2(x + z) + z^2(x + y))
\]
The Idiot’s Guide to Symmetric Inequalities

1. Homogenize
2. Multiply out all denominators, expand, and rewrite using symmetric notation.
Example of Bash

Example

(IMO 2005 P3) Let $x, y, z$ be three positive reals such that $xyz \geq 1$. Prove that

$$\frac{x^5 - x^2}{x^5 + y^2 + z^2} + \frac{y^5 - y^2}{x^2 + y^5 + z^2} + \frac{z^5 - z^2}{x^2 + y^2 + z^5} \geq 0$$
Example of Bash

Example

(IMO 2005 P3) Let \( x, y, z \) be three positive reals such that \( xyz \geq 1 \). Prove that

\[
\frac{x^5 - x^2}{x^5 + y^2 + z^2} + \frac{y^5 - y^2}{x^2 + y^5 + z^2} + \frac{z^5 - z^2}{x^2 + y^2 + z^5} \geq 0
\]

Solution.

Homogenizing and rearranging, it suffices to show

\[
3 \geq (x^3 yz + xy^3 z + x y z^3) \left( \frac{1}{x^5 + xy^3 z + xyz^3} + \frac{1}{x^3 yz + y^5 + xyz^3} + \frac{1}{x^3 yz + xy^3 z + z^5} \right)
\]

Multiply out and using symmetric notation, this is equivalent to

\[
\sum_{\text{sym}} x^1 y z + 4 \sum_{\text{sym}} x^7 y^5 + \sum_{\text{sym}} x^6 y^3 z^3 \geq 2 \sum_{\text{sym}} x^6 y^5 z + \sum_{\text{sym}} x^8 y^2 z^2 + \sum_{\text{sym}} x^5 y^5 z^2 + \sum_{\text{sym}} x^6 y^4 z^2
\]

Which follows from

\[
(10, 1, 1) + (6, 3, 3) \quad \geq \quad 2(8, 2, 2) \\
(8, 2, 2) \quad \geq \quad (6, 4, 2) \\
2(7, 5, 0) \quad \geq \quad 2(6, 5, 1) \\
(7, 5, 0) \quad \geq \quad (5, 5, 2) \\
(7, 5, 0) \quad \geq \quad (6, 4, 2)
\]