The Art of Counting

Bijections, Double Counting

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Today, we will consider some commonly-used paradigms of counting:

- Straightforward, careful counting
- Bijection
- Counting in multiple ways
Paradigm 1: Careful Straightforward Counting

- Comprehensive enumeration/case work
- Make sure to count every case
- Don’t double count
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**Sum Rule:**
If \( A = A_1 \cup A_2 \cup \cdots \cup A_n \), \( A_i \cap A_j = \emptyset \), then

\[
|A| = |A_1| + |A_2| + \cdots + |A_n|
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**Product Rule:**
If $W = W_1 \times W_2 \times \cdots \times W_n$ (Cartesian set product), then

$$|W| = |W_1||W_2| \cdots |W_n|$$
Basic Tool: Binomial Coefficients

How many subsets of \( \{1, 2, \ldots, n\} \) are there with exactly \( m \) elements?
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\(n\) ways to choose 1st element, \(n - 1\) ways to choose 2nd, \(\cdots\). So \(n(n - 1)\cdots(n - m + 1)\)?
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\[
\binom{n}{k} = \frac{n(n-1)\cdots(n-m+1)}{m(m-1)\cdots1} = \frac{n!}{(n-m)!m!}
\]
Basic Tool: Inclusion-Exclusion Principle

If we know $A = A_1 \cup \cdots \cup A_n$ but the $A_i$'s are not necessarily disjoint, how do we count $|A|$?

Example: suppose $|A| = 9$, $|B| = 6$, $|C| = 9$, $|A \cap B| = 3$, $|A \cap C| = 4$, $|B \cap C| = 2$, $|A \cap B \cap C| = 1$, what is $|A \cup B \cup C|$?

$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$

$= 9 + 6 + 9 - 3 - 4 - 2 + 1 = 16$
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|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|
\]
\[
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\]
\[
= 16
\]

In general,

\[
|A_1 \cup \cdots \cup A_n| = \sum_i |A_i| - \sum_{i<j} |A_i \cap A_j| + \sum_{i<j<k} |A_i \cap A_j \cap A_k| \cdots
\]
Putting the theory into practice

Example 1

[Derangements] At a Secret Santa party, there are \( n \) guests, who each brings a present. Once all presents are collected, they are permuted randomly, and redistributed to the guests. What is the probability no guest receives his/her own gift? What does this converge to as \( n \to \infty \)?
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Solution.

Let the set of all permutations of $\{1, 2, \cdots, n\}$ be $U$. Let $D \subset U$ be the set of permutations without fixed points. We seek $\frac{|D|}{|U|}$.
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$$|A_1 \cup \cdots \cup A_n| = \sum_i |A_i| - \sum_{i<j} |A_i \cap A_j| \cdots$$

$$= (n-1)! - \binom{n}{2}(n-2)! + \binom{n}{3}(n-3)!$$

$$= n! \sum_{i=1}^{n} \frac{(-1)^{i+1}}{i!}$$
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$$|A_1 \cup \cdots \cup A_n| = \sum_i |A_i| - \sum_{i<j} |A_i \cap A_j| \cdots$$

$$= (n-1)! - \binom{n}{2}(n-2)! + \binom{n}{3}(n-3)!$$

$$= n! \sum_{i=1}^n \frac{(-1)^{i+1}}{i!}$$

The probability is

$$\frac{|D|}{|U|} = \sum_{i=0}^n \frac{(-1)^i}{i!} \to \frac{1}{e}$$
Paradigm 2: Constructing a Bijection

A bijection is a one-to-one correspondence:

\[ f: A \rightarrow B \]

where \( f \) is one-to-one (no two elements in \( A \) are mapped to the same in \( B \)) and onto (for every element in \( B \), some element in \( A \) maps to it).

Equivalently, \( f \) is a bijection if there is an inverse map:

\[ \exists g: B \rightarrow A, \text{s.t.} \forall a \in A, g(f(a)) = a. \]

We frequently show that two sets are equal in size by constructing a bijection.
**Paradigm 2: Constructing a Bijection**

A bijection is a one-to-one correspondence:

Given sets $A$, $B$, a bijection $f$ is $f : A \rightarrow B$ that is **one-to-one** (no two elements in $A$ are mapped to the same in $B$) and **onto** (for every element in $B$, some element in $A$ maps to it.)

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Simple Bijection

Example 2

(CMO 2005) Consider an equilateral triangle of side length \( n \), which is divided into unit triangles, as shown. Let \( f(n) \) be the number of paths from the triangle in the top row to the middle triangle in the bottom row, such that adjacent triangles in our path share a common edge and the path never travels up (from a lower row to a higher row) or revisits a triangle. An example of one such path is illustrated below for \( n = 5 \). Determine the value of \( f(2005) \).

\[
\text{Diagram of a triangle with paths}
\]
Simple Bijection

Solution.
We show that there is a bijective mapping between valid paths and ordered lists \((a_1, a_2, \cdots, a_n)\), where \(1 \leq a_i \leq i\). Essentially \(a_i\) indicates where the path “exit” the \(i\)th row and enter the \((i + 1)\)th row. For any valid path, this ordered lists exists. For any ordered list, we can reconstruct the path uniquely. The number of such ordered lists is exactly \(n!\), hence \(f(2005) = 2005!\).
More Involved Bijection

Example 3

(Catalan Numbers) In a \( n \times n \) grid, we draw rectilinear paths from \((0, 0)\) to \((n, n)\), going only in positive \(x\) and \(y\) directions. How many such paths are there that stay below the line \(y = x\)?

Answer:

\[
C_n = \binom{2n}{n} - \binom{2n}{n-1}
\]

Proof.

It suffices to show that the number of paths that cross the line is \(\binom{2n}{n-1}\).

But this is exactly the number of paths from \((0, 0)\) to \((n-1, n+1)\). So try to find a bijection.

A path crosses \(y = x\) iff it touches \(y = x+1\). Map \(f\): take the first time the path touches \(y = x+1\) and reflect the following subpath across \(y = x+1\). Inverse map: apply paths from \((0, 0)\) to \((n-1, n+1)\) touch \(y = x+1\). Take the first touch, and reflect the following subpath across \(y = x+1\). The maps are inverses because the first touch is preserved by both maps.
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(Catalan Numbers) In a $n \times n$ grid, we draw rectilinear paths from $(0, 0)$ to $(n, n)$, going only in positive $x$ and $y$ directions. How many such paths are there that stay below the line $y = x$?

Answer: $C_n = \frac{1}{n+1} \binom{2n}{n} = \binom{2n}{n} - \binom{2n}{n-1}$
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Simple Example

Example 4

15 students join a summer course. Every day, 3 students are on duty after school to clean the classroom. After the course, it was found that every pair of students has been on duty together exactly once. How many days does the course last for?

Solution.

We count the total number $P$ of pairs of students who work together for all days. (Pairs are considered different if the same pairing happens on different days.) There are $n$ days and each day there are 3 pairs. So $P = 3n$. On the other hand $P = \binom{15}{2}$. Hence, $n = \frac{1}{2}\binom{15}{2} = 35$. 
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Hence, $n = \frac{1}{2} \binom{15}{2} = 35$. 

\[ \square \]
Example 5

(CMO 2006) There are $2n + 1$ teams in a round-robin tournament, in which each team plays every other team exactly once, with no ties. We say that teams $X$, $Y$, $Z$ form a *cycle triplet* if $X$ beats $Y$, $Y$ beats $Z$ and $Z$ beats $X$. Determine the maximum number of cyclic triplets possible.

**Proof.**

Count the # of the following types of “angles”

![Diagram showing types A, B, and C angles]
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Proof.

Count the number of the following types of “angles”
Each cyclic triangle has 3 type C angles, while each non-cyclic triangle has 1 angle of each time.

```
Type A
```
```
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Proof.

Count the number of the following types of “angles”

Each cyclic triangle has 3 type C angles, while each non-cyclic triangle has 1 angle of each time.

\[
\begin{align*}
\text{(\# of non-cyclic triangles)} &= \frac{1}{2} \left[ (\text{\# of type A angles}) + (\text{\# of type B angles}) \right] \\
&= \frac{1}{2} \sum_{i=1}^{2n+1} \binom{a_i}{2} + \binom{2n-a_i}{2} \\
&\geq \frac{1}{2} \sum_{i=1}^{2n+1} \binom{n}{2} + \binom{n}{2} \\
&= \frac{n(n-1)(2n+1)}{2n+1} \\
\end{align*}
\]

Hence, the number of cyclic triangles is at least $\frac{n(n+1)(2n+1)}{6}$.

To show this bound can be attained, label the vertices 1, 2, \ldots, $2n + 1$ and put directed edge $i \rightarrow j$ iff $j - i \pmod{2n+1} \in \{1, 2, \ldots, n\}$. 
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$(\# \text{ of non-cyclic triangles})$

$= \frac{1}{2}[(\# \text{ of type A angles}) + (\# \text{ of type B angles})]$

$= \frac{1}{2} \sum_{i=1}^{2n+1} \binom{a_i}{2} + \binom{2n-a_i}{2}$

$\geq \frac{1}{2} \sum_{i=1}^{2n+1} \binom{n}{2} + \binom{n}{2}$

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Hence, the number of cyclic triangles is at least

$\binom{2n+1}{3} - \frac{n(n-1)(2n+1)}{2} = \frac{n(n+1)(2n+1)}{6}$. 
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& = \frac{1}{2} \sum_{i=1}^{2n+1} \binom{a_i}{2} + \binom{2n-a_i}{2} \\
& \geq \frac{1}{2} \sum_{i=1}^{2n+1} \binom{n}{2} + \binom{n}{2} \\
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Hence, the $\#$ of cyclic triangles is at least

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To show this bound can be attained, label the vertices $1, 2, \cdots, 2n + 1$ and put directed edge $i \to j$ iff $j - i (\text{ mod } 2n + 1) \in \{1, 2, \cdots, n\}$. \qed
Conclusion

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Enjoy problem set 3! All problems have nice solutions, so try not to brute force.
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