B Online Appendix

In this Online Appendix we provide formal statements and proofs of the claims made in Section 5 of “Bayesian Persuasion with Heterogenous Priors”, Alonso and Câmara (2015).

Consider the extended model with private priors described in Section 5. As an application of (30), consider the pure persuasion model from Section 4.3. When the sender knows the receiver’s prior, Proposition 5(i) provides conditions on the likelihood ratio of priors such that persuasion is valuable. Suppose that these conditions are met and the sender strictly benefits from providing experiment $\pi$ to a particular receiver. By a continuity argument, the same $\pi$ strictly benefits the sender when she faces another receiver whose beliefs are not too different. Consequently, even if the sender does not know the receiver’s prior, persuasion remains beneficial when the receiver’s possible priors are not too dispersed. Proposition B.1 provides an upper bound on how dispersed these beliefs can be. To this end, let $R$ be the set of likelihood ratios induced by the priors in the support of $h(p^R|p^S)$,

$$R = \{r^R : \{r^R_\theta = p^R_\theta / p^S_\theta\}_{\theta \in \Theta}, p^R \in \text{Supp}(h(p^R|p^S))\}.$$

(42)

Proposition B.1 Suppose that $r^R$ and $r^R_\theta$ are not collinear w.r.t. $W$ for all $r^R \in R$, and let $m = \frac{1}{2} \max_{a} \left| \frac{u'_S(a)}{u'_S(a)} \right| > 0$. If for all $r^R, r^{R'} \in R$

$$\|r^R - r^{R'}\| \leq \beta,$$

(43)

with $\beta$ given by (51), then the sender benefits from persuasion.

The condition on $r^R$ and $r^R_\theta$ implies that if the sender knew the receiver’s prior, then she could find an experiment with a positive value (cf. Proposition 5). The bound $\beta$ is defined below by (51), as a function of the curvature of $u_S$. From (43), $\beta$ represents a lower bound on the cosine of the angle between any two likelihood ratios in the support of $h(p^R|p^S)$. Therefore, (43) describes how different the receiver’s possible prior beliefs can be for the sender still to benefit from persuasion, by imposing an upper bound on the angle between any two likelihood ratios in $R$.

Proof: The proof of this Proposition will make use of the following lemma:
Lemma B.1 Let $R$ be defined by (42) and $m = \frac{1}{2} \max \{u_S^a(a)\} > 0$, and for each $r \in R$, define

$$\Delta_S = \frac{\langle q^S, r^R \theta \rangle}{\langle q^S, r^R \rangle} - \langle p^R, \theta \rangle,$$

and define $l_{r^S}(\varepsilon)$ as

$$l_{r^S}(\varepsilon) = \frac{\langle \varepsilon, r^R \rangle}{\Delta_S}. \quad (44)$$

For any $\varepsilon$ and $r^R \in R$ such that

$$l_{r^S}(\varepsilon) < -m \text{ and } \Delta_S > 0, \text{ with } p^S + \varepsilon \in \Delta(\Theta), \quad (45)$$

there exists an experiment $\pi$ with the following properties: (i) Some realization of $\pi$ induces in the sender the belief $p^S + \varepsilon$; and (ii) $\pi$ increases the expected utility of the sender when the receiver’s associated likelihood ratio is $r^R$.

Proof: The function $l_{r^S}(\varepsilon)$ has an immediate interpretation as a measure of disagreement: the numerator $\langle \varepsilon, r^R \rangle$ is the difference in the probability that the receiver and sender attach to a realization inducing a posterior $q^S = p^S + \varepsilon$ on the sender, divided by the probability that the sender ascribes to such realization, while the denominator is the change in the receiver’s action when the sender changes her belief to $q^S$. We first show that if some $\varepsilon$ satisfies (45), then the value of information control is positive. Consider $V_S$ defined in (12), which in this case can be written as

$$V_S(q^S) = u_S \left( \frac{\langle q^S, r^R \theta \rangle}{\langle q^S, r^R \rangle} \right),$$

with gradient at $p^S$

$$\nabla V_S(p^S) = u_S^\prime(\langle p^R, \theta \rangle) \left( r^R \theta - \langle p^R, \theta \rangle r^R \right).$$

By Corollary 1, the value of information control is positive if and only if there exists $\varepsilon$, with $p^S + \varepsilon \in \Delta(\Theta)$, such that

$$\langle \nabla V_S(p^S), \varepsilon \rangle < V_S(p^S + \varepsilon) - V_S(p^S). \quad (46)$$

We now show that an $\varepsilon$ satisfying (45) also satisfies (46). Since

$$u_S \left( \frac{\langle q^S, r^R \theta \rangle}{\langle q^S, r^R \rangle} \right) - u_S(\langle p^R, \theta \rangle) - u_S^\prime(\langle p^R, \theta \rangle) \left( \frac{\langle q^S, r^R \theta \rangle}{\langle q^S, r^R \rangle} - \langle p^R, \theta \rangle \right) = \int_{\langle p^R, \theta \rangle}^{\langle q^S, r^R \theta \rangle} \int_{\langle p^R, \theta \rangle}^{\langle q^S, r^R \rangle} u_S^\prime(\tau) d\tau dt,$$

we can rewrite (46) as

$$u_S^\prime(\langle p^R, \theta \rangle) \langle \varepsilon, r^R \rangle \Delta_S < \int_{\langle p^R, \theta \rangle}^{\langle q^S, r^R \theta \rangle} \int_{\langle p^R, \theta \rangle}^{\langle q^S, r^R \rangle} u_S^\prime(\tau) d\tau dt.$$
By the mean value theorem, we have

$$
\int_{(p^R, \theta)} \int_t^{p^S, \theta} u''_S(\tau) d\tau dt \geq -\max |u''_S(a)| \int_{(p^R, \theta)} \int_t^{p^S, \theta} d\tau dt = -\frac{1}{2} \max |u''_S(a)| \Delta_S^2.
$$

Moreover, if $\varepsilon$ satisfies (45), then it also satisfies

$$
\langle \varepsilon, r^R \rangle \min u'_S(a) < -\frac{1}{2} \max |u''_S(a)| \Delta_S,
$$

implying that $\varepsilon$ also satisfies (46) since

$$
u'_S(\langle p^R, \theta \rangle) \langle \varepsilon, r^R \rangle \Delta_S < \langle \varepsilon, r^R \rangle \Delta_S \min u'_S(a) < -\frac{1}{2} \max |u''_S(a)| \Delta_S^2 \leq \int_{(p^R, \theta)} \int_t^{p^S, \theta} u''_S(\tau) d\tau dt.
$$

For each $\varepsilon$ satisfying (45), we now construct an experiment that improves the sender’s expected utility and that has a realization that induces belief $p^S + \varepsilon$ in the sender. Let $\nu$ be the excess of the right-hand side over the left-hand side in (46),

$$
\nu = V_S(p^S + \varepsilon) - V_S(p^S) - \langle \nabla V_S(p^S), \varepsilon \rangle > 0. \quad (47)
$$

Consider the experiment $\pi(\varepsilon, \delta)$ with $Z = \{\varepsilon^+, \varepsilon^-\}$, such that $\Pr_S[z = \varepsilon^+] = \delta$, and if $z = \varepsilon^+$, then the sender’s posterior is $p^S + \varepsilon$. A Taylor series expansion of $V_S(q^S)$ yields

$$
V_S(q^S) = V_S(p^S) + \langle \nabla V_S(p^S), q^S - p^S \rangle + L \left(q^S - p^S\right), \text{ with } \lim_{t \to 0} \frac{L \left(t \left(q^S - p^S\right)\right)}{t} = 0. \quad (48)
$$

Then, the sender’s gain from $\pi(\varepsilon, \delta)$ is

$$
\Delta_{\pi(\varepsilon, \delta)} = \delta \left(V_S(p^S + \varepsilon) - V_S(p^S)\right) + (1 - \delta) \left(V_S(p^S - \frac{\delta}{1 - \delta} \varepsilon) - V_S(p^S)\right)
$$

$$
= \delta \left(\nu + \langle \nabla V_S(p^S), \varepsilon \rangle\right) - \delta \langle \nabla V_S(p^S), \varepsilon \rangle + L \left(-\frac{\delta}{1 - \delta} \varepsilon\right)
$$

$$
= \delta \left(\nu - (1 - \delta) \frac{L \left(-\delta \varepsilon / (1 - \delta)\right)}{(-\delta / (1 - \delta))}\right).
$$

The convergence to zero of the second term in the parentheses when $\delta$ tends to zero and $\nu > 0$ guarantees the existence of $\delta > 0$ such that $\Delta_{\pi(\varepsilon, \delta)} > 0$. ■

**Proof of Proposition B.1:** First, we introduce additional notation. With $l_{p^R}(\varepsilon)$ defined as in (44), define the sets $M(r^R)$ by

$$
M(r^R) = \{\varepsilon : l_{p^R}(\varepsilon) < -m, \Delta_S > 0, p^S + \varepsilon \in \Delta(\Theta)\}.
$$
Note that $r^S$ and $\theta$ are negatively collinear if and only if $r^R$ and $r^R\theta$ are positively collinear. That is, the condition on Proposition 5 could instead be stated in terms of collinearity of $r^R$ and $r^R\theta$. Moreover, if $r^R$ and $r^R\theta$ are not collinear, then the restriction of $l_{rR}(\varepsilon)$ to $\{\varepsilon : \langle \varepsilon, \lambda \rangle = 0\}$ is surjective, and, thus, the set $M(r^R)$ is non-empty.

Define the function

$$
\Psi(\varepsilon, r^R) = \langle \varepsilon, r^R - mf^R \rangle + \left(\langle \varepsilon, r^R \rangle \right)^2,
$$

which characterizes $M(r^R)$ since for $\varepsilon$ such that $p^S + \varepsilon \in \Delta(\Theta)$, $\Psi(\varepsilon, r^R) \leq 0$ and $\langle \varepsilon, f^R \rangle \geq 0$ if and only if $\varepsilon \in M(r^R)$. Finally, let

$$
\gamma = 2 \left(1 + m (\max |\theta| + \|\theta\|) + (4 + m \|\theta\|) \sup_{r^R \in R} \|r^R\| \right),
$$

$$
Z = \min_{\varepsilon \in \{p^S + \varepsilon \in \Delta(\Theta)\}, r^R \in R} \Psi(\varepsilon, r^R) \text{ s.t. } \langle \varepsilon, r^R \theta - \langle p^S, r^R \theta \rangle \rangle \leq 0, r^R \in R.
$$

Under the conditions of Proposition B.1, $Z < 0$. Finally, define $\beta$ in (43) as

$$
\beta = \frac{|Z|}{\gamma}.
$$

Our proof is structured in two steps that show (i) if $\cap_{r^R \in R} M(r^R)$ is non-empty, then following Lemma B.1 allows us to design an experiment $\pi$ that increases the sender’s expected utility for every receiver’s belief in the support of $h(p^R|p^S)$; and (ii) under the conditions of Proposition B.1, $\cap_{r^R \in R} M(r^R) \neq \emptyset$.

**Step (i)** - Suppose that $\varepsilon \in \cap_{r^R \in R} M(r^R)$. Consider $\underline{v}$ as defined by (47). As $\underline{v}$ is a continuous function of $r^R$ in the compact set $R$, it achieves a minimum $\underline{v} = \min_{r^R \in R} \underline{v} > 0$. Then, define $\hat{\delta}$ as

$$
\hat{\delta} = \min \left\{ \delta : \underline{v} + \frac{L \left(1 - \frac{\delta}{\delta} \varepsilon \right)}{\delta} \geq 0 \right\},
$$

with the function $L$ given by (48). Now, define the experiment $\pi(\varepsilon, \hat{\delta}')$ as in the proof of Lemma B.1—i.e., $Z = \{\varepsilon^+, \varepsilon^-\}, q^S(\varepsilon^+) = p^S + \varepsilon$ and $\Pr_S[z = \varepsilon^+] = \delta'$, and set $\delta' = \hat{\delta}$. Then, the sender’s gain from $\pi(\varepsilon, \hat{\delta}')$ is positive for any receiver’s prior in $\text{Supp}(h(p^R|p^S))$.

**Step (ii)** - Fix $p^R$ with associated likelihood ratio $r^R \in R$. For any $r^R \in R$ with $\eta = r^R - r^R'$, we have

$$
\Psi(\varepsilon, r^R) - \Psi(\varepsilon, r^R') = \left(1 + m \langle p^S, r^R' \theta \rangle + \langle \varepsilon, r^R + r^R' \rangle \right) \langle \varepsilon, \eta \rangle - m \langle \varepsilon, \eta \theta \rangle + m \langle p^S, \eta \theta \rangle \langle \varepsilon, r \rangle.
$$
The following bounds make use of the Cauchy-Schwartz inequality (in particular, the implication that $|\langle \varepsilon, \eta \theta \rangle| \leq \|\varepsilon\| \|\eta\| \|\theta\|$—see Steele, 2004)\textsuperscript{27} and the fact that $\|p^S\| \leq 1$ and $\|\varepsilon\| = \|q^S - p^S\| \leq 2$,

\[
\left| 1 + m \left\langle p^S, r^R \theta \right\rangle + \left\langle \varepsilon, r^R + r^R \right\rangle \right| \leq 1 + m \max_{r^R \in R} \|r^R\|, \\
|m \langle \varepsilon, \eta \theta \rangle| \leq m \|\varepsilon\| \|\eta\| \|\theta\| \leq 2m \|\eta\| \|\theta\|, \\
|m \langle p^S, \eta \theta \rangle \langle \varepsilon, r \rangle| \leq 2m \|\eta\| \|\theta\| \sup_{r^R \in R} \|r^R\|.
\]

From these bounds, we then obtain the following estimate

\[
\left| \Psi (\varepsilon, r^R) - \Psi (\varepsilon, r^R) \right| \leq \left| 1 + m \left\langle p^S, r^R \theta \right\rangle + \left\langle \varepsilon, r^R + r^R \right\rangle \right| \|\varepsilon\| \|\eta\| \\
+ \left| m \langle \varepsilon, \eta \theta \rangle \right| + \left| m \langle p^S, \eta \theta \rangle \langle \varepsilon, r \rangle \right| \\
\leq 2 \left( 1 + m \max_{r^R \in R} \|r^R\| \right) \|\eta\| + 2m \|\theta\| \|\eta\| \\
+ 2m \|\theta\| \sup_{r^R \in R} \|r^R\| \|\eta\| \\
= \gamma \|\eta\|,
\]

where $\gamma$ is defined by (49). Selecting $\varepsilon'$ an $r^R$ that solve the program (50) and noting that $Z < 0$, we have that for any $r^R \in R$,

\[
\Psi (\varepsilon', r^R) = \Psi (\varepsilon', r^R) + \Psi (\varepsilon', r^R) - \Psi (\varepsilon', r^R) \leq Z + \gamma \|\eta\| \leq Z + |Z| = 0.
\]

This implies that $\varepsilon' \in M(r^R)$ for all $r^R \in R$. \blacksquare