Optimal order placement in a limit order book

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Outline

1. Background: Algorithm trading in different time scales
2. Some note on optimal execution in daily trading
3. Optimal execution with key statistics of the limit order book
   - Basics of LOB
   - Model without price impact
   - Model with linear price impact
4. Optimal strategy with full information of the limit order book
What is algorithm trading

- In an electronic order-driven market, orders arrive to the exchange and wait in the LOB (Limit Order Book) to be executed.
- LOB contains detailed information of each order: size, buy and sell type (market order or limit order).
- Algorithm trading: automatic and rapid trading of large quantities based on LOB information, orders are specified and implemented by computer algorithm.
Order flow: traffic is heavy...

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<tr>
<td>Citigroup</td>
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**Table:** Average number of orders in a 10 second interval (June 26th, 2008)

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<td>General Motors</td>
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**Table:** Number of changes in the price on June 26th, 2008.
On main issue in algorithm trading

Selling/buying a (large) number of shares

- Trading strategy has impact on stock price
  - Too quick selling will cause price to drop too fast
  - Too slow selling will loss appropriate appreciation

- Transaction cost
Algorithm trading in three time scales

<table>
<thead>
<tr>
<th>Regime</th>
<th>Time scale</th>
<th>Issues</th>
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</thead>
<tbody>
<tr>
<td>Ultra-high frequency (UHF)</td>
<td>Tick ($\sim 10^{-3} - 1$ s)</td>
<td>Microstructure, Latency</td>
</tr>
<tr>
<td>High Frequency (HF)</td>
<td>$\sim 10 - 10^2$ s</td>
<td>Optimal placement (***)</td>
</tr>
<tr>
<td>“Daily”</td>
<td>$\sim 10^3 - 10^4$ s</td>
<td>Trading, Option hedging (*)</td>
</tr>
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</table>

**Table:** A hierarchy of time scales.
Optimal execution in algorithmic trading

- On a daily (weekly) time scale, give the optimal rate of trading:
- On a 1/5 minute time scale, optimally place the orders in the limit order book in order to get the best price and minimize trading costs
Optimal execution (on daily) time scale

- Brownian motion/random walk based models
- Two types of price impact: temporary vs permanent
- Finite time horizon, with 0 discount rate
- Static/deterministic trading is optimal

VWAP, POV, Bertimas and Lo (98), Almgren-Chriss (99,01,03), Huberman and Stanzl (00), Obizhaeva and Wang (05), Schied and Schoneborn (07,08), Almgren and Lorenz (07), Gatheral (2010)
Why deterministic is optimal

- Because of the martingale property, minimize the price impact is equivalent to minimizing the cost
- Because of the martingale property, minimizing the price impact is equivalent to a deterministic optimization problem
- No known explicit solution for other existing models with price impact

Gatheral (2010), Predoiu, Shaiket, and Shreve (2010)
A simple example for which optimal strategy is not static (Guo and Zervos (2010))

- $Y_t(\omega)$: the number of shares held at time $t$, $Y_0 = y$
- $\xi_t(\omega)$: the total number of shares sold up to time $t$, is a non-decreasing process, so that
  \[ Y_t = y - \xi_t, \quad dY_t = -d\xi_t \]
- $S_t(\omega)$: the stock price at time $t$ with $S_t = e^{X_t}$
  \[ dX_t = \mu dt - \lambda d\xi_t + \sigma dW_t \]
  for some constant $\mu, \lambda, \sigma$, with $X = X_0 = \ln S_0$
Optimization problem

\[ V(x, y) = \sup_{\xi} E\left[ \int_0^T e^{-\delta t} (e^{X_t} - C) \ast d\xi_t \right] \]

with

- \( C \) the “transaction cost” of selling each share
- \( \delta \) the “impatience” index
- Non-linear price impact in the performance criterion

\[ (e^{X_t} - C) \ast d\xi_t = e^{X_t} d\xi_t^c + \int_0^{\Delta\xi_t} e^{X_t - \lambda u} du - Cd\xi_t \]
Explicit solution for $T = \infty$

- The value function is finite iff $\mu < \delta$.
- If $C = 0$ and $\mu < \delta$, sell off everything immediately is optimal.
- In general, when $C > 0$ and $\mu < \delta$:
  - The state space $\{(x, y), y > 0\}$ is divided into Continuation Region $\mathcal{C}$ and Action Region $\mathcal{A}$, and the free boundary is $x = F(y)$.
  - In $\mathcal{A}$, $e^x - C - \lambda W_x(x, y) - W_y(x, y) = 0$.
  - In $\mathcal{C}$, $\frac{1}{2} \sigma^2 W_{xx}(x, y) + \mu W_x(x, y) - \delta W(x, y) = 0$. 

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Derivation of solution

- Standard Verification Theorem Approach
- In $\mathcal{C}$, smooth-fit principle solves explicitly $F(y) = \ln \frac{nC}{n-1}$ with $n > 1$ satisfying $\frac{1}{2}\sigma^2 n^2 + \mu n - \delta = 0$
- In $\mathcal{A}$, two possible actions
  - Sell off everything immediately
  - Jump to the boundary and continue afterwards
- Value function is $C^2$. 
Algorithmic trading
Optimal placement with LOB
Optimal execution (on daily) time scale

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A limit buy order

![Order Book Diagram]

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A limit buy order

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A market sell order

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Order Book

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A cancelation of sell order
What is a LOB

- LOB contains detailed information of buy and sell types (market order or limit order) and their sizes
- Limit buy/sell order: an order to buy/sell a certain quantity at a given price
- Market buy/sell order: an order to buy/sell a certain quantity at the best available price in LOB
- (Best) ask price: lowest price in limit sell order
- (Best) bid price: the highest price in limit buy order
Main characteristics of LOB

- Ask/bid prices in high frequency trading are believed by practitioners to be mean-reverting.
- Probability of a limit order being executed depends on the queue length, its position in the LOB, frequency of price changes, cancelation of orders...
- Traders face trade-off between limit orders and market orders: pay the spread and fees vs execution/inventory risk.
Our work (de larrard and Guo (2010))

- Address the following questions: should one use market orders or limit orders? At which levels should one place orders and how?
- Provide optimal order execution strategy with the incorporation of
  1. key statistics of the limit order book
  2. statistics and realistic assumptions on the order flow (arrival rates of market, limit orders and cancelation)
A simple “mean-reverting” model

- Fix a time horizon \([0, T]\).
- The spread between the bid price \((B_t, t \geq 0)\) and the ask price \((A_t, t \geq 0)\) is constant, say equal to one tick.
- The (ask) price \((A_t, t \geq 0)\) follows a continuous time, piecewise constant process.
- Increments of the ask are +1 or −1 tick.
- The probability of execution if spent time \(t\) as the first limit is \(F(t)\).
- The price moves at time \(0 \leq T_1 < T_2 < \cdots T_n \leq T\) and the distribution of the number of price moves before time \(T\) is given by a distribution \(G_T\).
Given $G_T = n$, $A_T = \sum_{i=1}^{n} X_i$ where

$$P(X_1 = 1) = P(X_1 = -1) = \frac{1}{2},$$

$$P(X_i = X_{i-1} | X_{i-1}) = p_c,$$

where $p_c$ is a constant

Note: when $p_c = \frac{1}{2}$, the price is a time-changed random walk; when $p_c < \frac{1}{2}$, the price “mean-reverts”.
Objective

- Buy $N$ orders before time $T > 0$ ($T \approx 1/5$ minutes)
- Split the $N$ orders in $(N_0, N_1, \ldots)$ where $N_0$ is the number of orders at market orders, $N_1$ the number of orders at the bid...
- If the $N$ orders are not executed at time $T$, have to buy the non-executed orders at the market price
- When one limit order is executed, the market gives a rebate of $r > 0$
- When the trader submits a market order, there is a fee of $f > 0$
- Given $(N_0, N_1, \ldots, N_k)$, find the optimal strategy for

$$\min_{(N_0, \ldots, N_k \ldots)} \sum_{k=0}^{\infty} N_k = N \mathbb{E}[C(N_0, \ldots, N_k, \ldots)]$$
Two key quantities for the model

\[ P\left(\sum_{i=1}^{n} X_i = k\right) = u(n, k), \]

\[ \phi(k, T) = \mathbb{P}\left[ \min_{0 \leq t \leq T} A(t) = k \right]. \]
Theorem

Given \( n \) price changes,

\[
u(n + k, n - k) = \sum_{i=1}^{2k} \left( \frac{1 - p_c}{2} \right)^i p_c^{k+n-i-1} g(n, k, i)
\]

where

\[
g(n, k, i) = \begin{cases} 
2 \binom{n-1}{i-1/2} \binom{k-1}{i-1/2}, & \text{n is odd} \\
\binom{n-1}{[(i-1)/2]} \binom{k-1}{[(i+1)/2]} + \binom{n-1}{[(i+1)/2]} \binom{k-1}{[(i-1)/2]}, & \text{n is even}
\end{cases}
\]
calculating $u(n + k, n - k)$ amounts to calculating the probability of a sequence of 1 and -1 of length $n+k$, with $n$ 1’s and $k$ -1’s and $i$ switches, with $i = 1, \cdots, 2k$. 

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For the second quantity, note that

\[ P[\min_{s \leq T} A_r = k | G_T = n] \]

can be calculated using the idea of the reflection principle of a random walk.
Derivation of optimal strategy: two steps

Consider $C_i(\gamma, T)$ the average cost of 1 share put at $i$-th limit from the ask, then

$$
\mathbb{E}[C(N_0, N_1, \ldots, N_k)] = \sum_{k=0}^{\infty} C_i(\gamma, T) N_i
$$

Key lemma

For all $i \geq 1$, there is a function $(f_i(\gamma, T), 0 \leq \gamma, \infty)$ such that $C_i(\gamma, T) = C_i(0, T) + f_i(\gamma, T)(C_i(\infty, T) - C_i(0, T))$. Moreover if $\forall k \geq 0, \phi(k, T) = \mathbb{P}[\min_{0 \leq t \leq T} A(t) = k],$

$$
C_i(0, T) = f - (r + f) \sum_{k=i}^{\infty} \phi(-k, T), \text{ and }
$$

$$
C_i(\infty, T) = f - (r + f + 1) \sum_{k=i-1}^{\infty} \phi(-k, T)
$$
Corollary

For all $0 \leq \gamma \leq \infty$ and for all $T > 0$,

$$C_0(\gamma, T) > ... > C_k(\gamma, T) > C_{k-1}(\gamma, T) > ... > C_1(\gamma, T)$$
Optimal dynamic strategy

Using the Dynamic Programming Principle, we have

**Theorem**

The optimal strategy is to 'peg' the price. That is, follow the price at the bid.
Estimating $G_T$ from LOB

Two possible candidates for $G_T$

- Empirical studies suggest that $G_T$ might be exponentially distributed
- Modeling the dynamics of order book $(q^a, q^b)$ by a correlated Brownian motion as in Cont and Larrard (2011), so that the distribution of next price change can be explicitly computed by Iryengar (1985).
A model with linear price impact

- Fix time $[0, T]$ and $0 = t_0 < t_1 < \cdots t_n = T$, $t_k - t_{k-1}$ constant
- Buy $N$ orders by time $T$. Orders are placed only at $0 = t_0 < t_1 < \cdots t_n = T$
- $A_t$ is the “mean-reverting” random walk as before, if no order placed
- $N^m_t, N^l_t$ the number of shares to be placed on market order and limit order at time $t$ respectively
- Linear price impact: additional price increase by $\alpha (N^m_t + N^l_t)$ with market order and (best) limit order placed
- Probability of limit order being executed by the next time step is a constant $p(t_k - t_{k-1})$
Remark on linear price impact

Recently an empirical study by Cont, Kubanov and Stoikov (2010) shows that on the trade level all trades have linear price impact.
The objective is to minimize

$$E\left[ \sum_{t=1}^{T} g(A_t, N_t^{m}, N_t^{l}, r, f) \right]$$

overall all sequences \( \{(N_1^{m}, N_1^{l}), (N_2^{m}, N_2^{l}), \cdots, (N_T^{m}, N_T^{l})\} \). Here \( g(X_t, N_t^{m}, N_t^{l}, r, f) \) is the cost at time \( t \) associated with placing orders \( N_t^{m}, N_t^{l} \), with fee \( f \) and spread for each marker order and rebate \( r \) for each limit order. In particular, one is to minimize the expected price.
Optimal dynamic strategy

Theorem

There exists an optimal sequence of \( \{(N_t^m, N_t^l), t = 0, \ldots, T\} \) which is solved explicitly and recursively through a quadratic programming with linear constraint. Depending on the parameters, there are three cases:

- Put everything on market order at time 0 or time \( T \)
- Put everything on limit order in a VWAP sense
- Put everything on limit order, but not in VWAP sense.

Note: Because the underlying price is not martingale, minimize the cost is in general not equivalent to minimizing the price impact.
Key idea for the underlying model

- Focus on the (best) bid and ask and their queue length
- The queue length is “renewed” according to some distributions after a price increase/decrease
Price dynamics

Figure: Joint density of bid and ask queues after a price move.
Optimal placement with full inclusion of LOB dynamics

- Take the ask and bid dynamics as a queuing system suggested in Cont and de Larrard (2011)
- Impulse control formulation of order placement (on both market order and limit orders)
- Associated HJB equation is with non-standard boundary conditions

→ Work in Progress
Related references

- Xin Guo and Mihail Zervos (2010) Optimal Selling With Liquidation Constraint and Transaction Cost
THANK YOU