Multivariate utility maximization with proportional transaction costs and random endowment

– Joint work with G. Benedetti (CREST, Dauphine) –

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Introduction I : Formulation of the problem

A concave function $U : \mathbb{R}^d \to [-\infty, \infty)$ is called a utility function supported on $\mathbb{R}_+^d$ if

- $C_U := \text{cl}(\text{dom}(U)) = \text{cl}\{x : U(x) > -\infty\} = \mathbb{R}_+^d$ and
- $U$ is increasing with respect to $\mathbb{R}_+^d$-(partial) order.

Consider the following problem

$$u(\mathcal{E}) := \sup\{\mathbb{E} [U(X + \mathcal{E})] : X \in \mathcal{A}_T^0\}$$

where

- $\mathcal{A}_T^0$ is the set of all final gains attainable from a zero initial portfolio (to be defined later).
- $\mathcal{E}$ is a (possibly unbounded) random endowment.
Introduction I : Formulation of the problem

A concave function \( U : \mathbb{R}^d \rightarrow [-\infty, \infty) \) is called a utility function supported on \( \mathbb{R}^d_+ \) if

- \( C_U := \text{cl}(\text{dom}(U)) = \text{cl}\{x : U(x) > -\infty\} = \mathbb{R}^d_+ \) and
- \( U \) is increasing with respect to \( \mathbb{R}^d_+(\text{partial}) \) order.

Consider the following problem

\[
\begin{align*}
u(\mathcal{E}) := \sup \{ \mathbb{E} [U(X + \mathcal{E})] : X \in A_T^0 \}
\end{align*}
\]

where

- \( A_T^0 \) is the set of all final gains attainable from a zero initial portfolio (to be defined later).
- \( \mathcal{E} \) is a (possibly unbounded) random endowment.
Introduction II: References

Max utility with TC:
- HJB approach: Davis-Norman (1990), Shreve-Soner (1994).
- Shadow prices: recent papers by Gerhold, Kallsen, Mühle-Karbe, Schachermayer.

Utility-based pricing with TC:
Introduction II: References

- Max utility with TC:
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- Utility-based pricing with TC:
TC Model: Bid-ask matrices

Main features of the model: All is measured in physical units/quantities, \( d \geq 1 \) risky assets (e.g., foreign currencies), the terms of trading are given by a **bid-ask process**

\[
\{\Pi_t(\omega), t \in [0, T]\} = \{(\Pi_{ij}^t(\omega))_{i,j=1}^d, t \in [0, T]\}
\]

an adapted, càdlàg, \( d \times d \) matrix-valued process s.t.
- \( \Pi_{ij} > 0, 1 \leq i, j \leq d \)
- \( \Pi_{ii} = 1, 1 \leq i \leq d \)
- \( \Pi_{ij} \leq \Pi_{ik}^j \Pi_{kj}^i, 1 \leq i, j, k \leq d \)

**Meaning:** To buy 1 unit of currency \( j \) one has to pay \( \Pi_{ij}^t(\omega) \) units of \( i \) (at time \( t \) when the state of world is \( \omega \))
TC Model: Solvency cones & consistent price systems

- **solvency cone**: $K_t = \text{cone}\{e^i, \Pi_{ij}^t e^i - e^j : 1 \leq i, j \leq d\}$
- $-K_t$ is the cone of portfolios available at price 0
- polar of $-K_t$: $K_t^* = \{w \in \mathbb{R}^d : \langle v, w \rangle \geq 0, \forall v \in K_t\}$
- **Financial interpretation**: $w \in K_t^*$ iff $w \in \mathbb{R}^d$ and $\Pi_{ij}^t w^i \geq w^j$
  \[ \Rightarrow \Pi_{ij}^t \geq \frac{w^j}{w^i} \Rightarrow \Pi_{ij}^t = (1 + \lambda_{ij}^t) \frac{w^j}{w^i} \text{ for some } \lambda_{ij}^t \geq 0 \]
- Every $w \in K_t^*$ (resp. in its interior) is called consistent (resp. strictly consistent) price system.
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- **solvency cone**: $K_t = \text{cone}\{ e^i, \prod_{t}^{ij} e^i - e^j : 1 \leq i, j \leq d \}$
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- **polar of $-K_t$**: $K^*_t = \{ w \in \mathbb{R}^d : \langle v, w \rangle \geq 0, \forall v \in K_t \}$
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- Every $w \in K_t^*$ (resp. in its interior) is called **consistent** (resp. **strictly consistent**) price system.
TC Model: Solvency cones & consistent price systems

- **solvency cone**: $K_t = \text{cone}\{ e^i, \Pi_{ij}^t e^i - e^j : 1 \leq i, j \leq d \}$
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- Every $w \in K_t^*$ (resp. in its interior) is called consistent (resp. strictly consistent) price system.
TC Model: Strictly consistent price processes

$Z$ is a random dynamic analogue of SCPS’s.

**Definition**

An $\mathbb{R}^d_+ \setminus \{0\}$-valued, adapted process $Z$ is a strictly consistent price process if

- is a càdlàg martingale
- If $Z_\tau \in \text{int}K^*_\tau \forall \tau$ stopping time, and $Z_{\sigma^-} \in \text{int}K^*_{\sigma^-} \forall \sigma$ predictable stopping time.

Relation with the usual concept of EMM: choose a numéraire $Z^1$, define $S_t = (1, Z^2_t / Z^1_t \ldots Z^d_t / Z^1_t)$ and set $d\mathbb{Q}/d\mathbb{P} = Z^1_T / Z^1_0$, then $S$ is a $\mathbb{Q}$-martingale.
TC Model: Admissible portfolios

Let $\Pi_t$ be a given Bid-Ask process. A $d$-dim process $X$ is an admissible self-financing portfolio process if

- is predictable and finite variation (may have left as well as right jumps!)
- $dX_t \in -K_t$, more precisely:

$$X_\tau - X_\sigma \in -K_{\sigma,\tau} = -\text{conv}(\bigcup_{\sigma \leq u < \tau} K_u, 0)$$

- is “bounded from below” by some threshold

*Interpretation:* $X^i_t = \text{quantity}$ of asset $i$ held by the agent in $t$. We denote $A^x$ the set of all admissible portfolio processes $X$ with $X_0 = x$, and $A^x_T := \{X_T : X \in A^x\}$.
TC Model: Super-replication theorem

Assumption

No-Arbitrage condition (NA): There exists a SCPP $Z$.

Let $Y \in L^0(\mathbb{R}^d, \mathcal{F}_T)$ a contingent claim such that $Y \succeq_T -a1$ (i.e. $Y + a1 \in K_T$) for some $a > 0$.

Theorem (C.-Schachermayer, 2006)

Under NA, the following sets are equal:

1. $\{x \in \mathbb{R}^d : \exists X \in \mathcal{A}^x, X_T \succeq Y\}$
2. $\{x \in \mathbb{R}^d : \langle Z_0, x \rangle \geq E[\langle Z_T, Y \rangle], \forall Z \in \mathcal{Z}^s\}$

For more info Kabanov-Safarian’s book (2010).
Let us come back to max problem

**Assumption**

*N- Arbitrage condition: There exists a SCPP Z*

Let $U$ denote a utility function such that $C_U = \mathbb{R}_+^d$. Our objective is

$$u(\mathcal{E}) := \sup\{\mathbb{E}[U(X + \mathcal{E})] : X \in \mathcal{A}_T^0\}.$$  

Recall from C.-Owen (2008) that $u(x)$ finite if $x \in \text{cl}(\mathcal{A}_T^0 \cap \mathbb{R})$, so we assume

$$x' \preceq \mathcal{E} \preceq x'' + X'', \quad x', x'' \in \text{int}(\mathcal{A}_T^0 \cap \mathbb{R}), X'' \in \mathcal{A}_T^0.$$  

We call $\mathcal{O}$ the set of random endowments with such properties.
Inada’s conditions I: Essential smoothness $U'(0) = \infty$

For proving the main results we need multivariate Inada’s conditions:

- **Essentially smoothness** (analogue of $U'(0) = \infty$)
- **Asymptotic satiability** (analogue of $U'(\infty) = 0$)

**Definition (Rockafellar’s Convex Analysis)**

A utility function $U : \mathbb{R}^d \rightarrow [-\infty, \infty)$ with $C_U = \mathbb{R}_+^d$ is said to be **essentially smooth** if

1. $U$ is differentiable in the interior of $\mathbb{R}_+^d$;
2. $\lim_{i \to \infty} |\nabla U(x_i)| = +\infty$ for any sequence $x_i \in \mathbb{R}_+^d$ converging to a boundary point of $\mathbb{R}_+^d$. 
Inada’s conditions II: Asymptotic satiability $U'(\infty) = 0$

- Let $U$ be a utility function, and let $C_U$ be its support cone. We say that a utility function $U$ is *asymptotically satiable* if given any $\epsilon > 0$ there exists an $x \in \text{dom}(U)$ such that

$$\partial U(x) \cap [0, \epsilon)^d \neq \emptyset.$$ 

- Recall that the dual function of $U$ is defined by

$$U^*(x^*) = \sup_{x \in \mathbb{R}^d} \{ U(x) - \langle x, x^* \rangle \}$$

- One can prove that asympt. satiability of $U$ is equivalent to $0 \in C_{U^*} := \text{cl} (\text{dom}(U^*))$. 

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Multivariate utility maximization with proportional transaction costs and random endowment
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Multivariate utility maximization with proportional transaction costs and random endowment
Duality I : Dual variables

Let $\mathcal{E} \in \mathcal{O}$. We look at the corresponding abstract problem and use the duality $(L^\infty, \text{ba})$.

- Set $\mathcal{C} = \mathcal{A}_T^0 \cap L^\infty(\mathbb{R}^d)$ and define
  \[
  \mathbb{U}_\mathcal{E}(X) := \mathbb{E} [U(X + \mathcal{E})].
  \]

  Then $\sup_{X \in \mathcal{C}} \mathbb{U}_\mathcal{E}(X) = u(\mathcal{E}) = \sup_{X \in \mathcal{A}_T^0} \mathbb{E} [U(X + \mathcal{E})]$.

- The dual of $\mathbb{U}_\mathcal{E}$ is given by
  \[
  \mathbb{U}_\mathcal{E}^*(m) := \sup_{X \in L^\infty(\mathbb{R}^d)} [\mathbb{U}_\mathcal{E}(X) - m(X)].
  \]

- Define the dual cone of $\mathcal{C}$ by
  \[
  \mathcal{D} := \{ m \in \text{ba}(\mathbb{R}^d; \mathbb{P}) : m(X) \leq 0 \quad \forall X \in \mathcal{C} \}.
  \]
Duality I : Dual variables

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- The dual of $U_\mathcal{E}$ is given by
  \[
  U^*_\mathcal{E}(m) := \sup_{X \in L^\infty(\mathbb{R}^d)} [U_\mathcal{E}(X) - m(X)].
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- Define the dual cone of $\mathcal{C}$ by
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  \]
Duality I: Dual variables

Let $\varepsilon \in \mathcal{O}$. We look at the corresponding abstract problem and use the duality $(L^\infty, ba)$.

- Set $\mathcal{C} = \mathcal{A}_T^0 \cap L^\infty(\mathbb{R}^d)$ and define

$$U_\varepsilon(X) := \mathbb{E} [U(X + \varepsilon)].$$

Then $\sup_{X \in \mathcal{C}} U_\varepsilon(X) = u(\varepsilon) = \sup_{X \in \mathcal{A}_T^0} \mathbb{E} [U(X + \varepsilon)].$

- The dual of $U_\varepsilon$ is given by

$$U_\varepsilon^*(m) := \sup_{X \in L^\infty(\mathbb{R}^d)} [U_\varepsilon(X) - m(X)].$$

- Define the dual cone of $\mathcal{C}$ by

$$\mathcal{D} := \{ m \in ba(\mathbb{R}^d; \mathbb{P}) : m(X) \leq 0 \quad \forall X \in \mathcal{C} \}. $$
Duality II

- Pick w.l.o.g. $X \in \mathcal{X}_T$ and $\mathcal{E} \in \mathcal{O}$ such that $X + \mathcal{E} \in \mathbb{R}_+^d$
- For any $m = m^c + m^p \in \mathcal{D}$ one has

$$U(X + \mathcal{E}) \leq U^* \left( \frac{dm^c}{d\mathbb{P}} \right) + \left\langle X + \mathcal{E}, \frac{dm^c}{d\mathbb{P}} \right\rangle$$

- Taking expectation, one has

$$E \left[ U(X + \mathcal{E}) \right] \leq E \left[ U^* \left( \frac{dm^c}{d\mathbb{P}} \right) + \left\langle X + \mathcal{E}, \frac{dm^c}{d\mathbb{P}} \right\rangle \right]$$

$$\leq E \left[ U^* \left( \frac{dm^c}{d\mathbb{P}} \right) \right] + m(\mathcal{E})$$

- Moreover: $U^*_\mathcal{E}(m) = E \left[ U^* \left( \frac{dm^c}{d\mathbb{P}} \right) \right] + m(\mathcal{E})$ for $m \in \text{ba}(\mathbb{R}_+^d)$
- so that $u(\mathcal{E}) \leq \inf_{m \in \mathcal{D}} U^*_\mathcal{E}(m)$. 

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Multivariate utility maximization with proportional transaction costs and random endowment
Duality II

- Pick w.l.o.g. $X \in \mathcal{A}_T$ and $\mathcal{E} \in \mathcal{O}$ such that $X + \mathcal{E} \in \mathbb{R}_+^d$
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- Taking expectation, one has

$$\mathbb{E} \left[ U(X + \mathcal{E}) \right] \leq \mathbb{E} \left[ U^* \left( \frac{dm^c}{d\mathcal{P}} \right) + \left\langle X + \mathcal{E}, \frac{dm^c}{d\mathcal{P}} \right\rangle \right]$$

$$\leq \mathbb{E} \left[ U^* \left( \frac{dm^c}{d\mathcal{P}} \right) \right] + m(\mathcal{E})$$

- Moreover: $U^*_{\mathcal{E}}(m) = \mathbb{E} \left[ U^* \left( \frac{dm^c}{d\mathcal{P}} \right) \right] + m(\mathcal{E})$ for $m \in \text{ba(}\mathbb{R}_+^d\text{)}$
- so that $u(\mathcal{E}) \leq \inf_{m \in \mathcal{D}} U^*_{\mathcal{E}}(m)$. 

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- For any $m = m^c + m^p \in \mathcal{D}$ one has
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  - Moreover: $U^*_\mathcal{E}(m) = \mathbb{E} \left[ U^* \left( \frac{dm^c}{d\mathbb{P}} \right) \right] + m(\mathcal{E})$ for $m \in \text{ba}(\mathbb{R}^d_+)$.
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Duality III: No duality gap and existence

Theorem (Lagrange Duality Theorem)

If $\mathcal{E} \in \mathcal{O}$ then

$$\sup_{X \in \mathcal{C}} \mathcal{U}_\mathcal{E}(X) = u(\mathcal{E}) = \min_{m \in \mathcal{D}} \mathcal{U}_\mathcal{E}^*(m) \in \mathbb{R}.$$ 

Let $\hat{m} = \hat{m}_c + \hat{m}_p \in \mathcal{D}$ denote the minimizer, which is unique if $U$ is strictly concave.

Moreover, if $\mathcal{E} \in \mathcal{O} \cap L^\infty(\mathbb{R}^d)$ and $x \mapsto u(x + \mathcal{E})$ is asymptotic satiable, then the optimizer $\hat{X}$ exists and

$$\hat{X} = (\nabla U)^{-1} \left( \frac{d \hat{m}_c}{d\mathcal{P}} \right) - \mathcal{E}.$$
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Let \( \hat{m} = \hat{m}^c + \hat{m}^p \in \mathcal{D} \) denote the minimizer, which is unique if \( U \) is strictly concave.

Moreover, if \( \mathcal{E} \in \mathcal{O} \cap L^\infty(\mathbb{R}^d) \) and \( x \mapsto u(x + \mathcal{E}) \) is asymptotic satiable, then the optimizer \( \hat{X} \) exists and

\[
\hat{X} = (\nabla U)^{-1} \left( \frac{d\hat{m}^c}{d\mathbb{P}} \right) - \mathcal{E}.
\]
Sufficient conditions for existence

- The asymptotic satiability of $u(\mathcal{E} + x)$ is difficult to check. Nonetheless ...

- Let $U^*$ satisfy the growth condition

$$U^*(\epsilon x^*) \leq \zeta(\epsilon)(U^*(x^*)^+ + 1)$$

for all $x^* \in \mathbb{R}^d_+, \epsilon \in (0, 1]$ and for some positive function $\zeta$. Then, both $U$ and $x \mapsto u(\mathcal{E} + x)$ are asymptotically satiable if $\mathcal{E} \in O \cap L^\infty$.

- As in the case with no random endowment (see C.-Owen), a suitable notion of RAE with more concavity (Multivariate Risk Aversion) implies the growth condition for $U^*$.

- Liquidation to, e.g., the first asset can be included in the picture taking $U(x_1 \ldots x_d) = U(x_1)$ (as in C.-Owen).
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- Let \( U^* \) satisfy the growth condition

\[
U^*(\varepsilon x^*) \leq \zeta(\varepsilon)(U^*(x^*)^+ + 1)
\]

for all \( x^* \in \mathbb{R}^d_{++} \), \( \varepsilon \in (0, 1] \) and for some positive function \( \zeta \).

Then, both \( U \) and \( x \mapsto u(\mathcal{E} + x) \) are asymptotically satiable if \( \mathcal{E} \in \mathcal{O} \cap L^\infty \).

- As in the case with no random endowment (see C.-Owen), a suitable notion of RAE with more concavity (Multivariate Risk Aversion) implies the growth condition for \( U^* \).
- Liquidation to, e.g., the first asset can be included in the picture taking \( U(x_1 \ldots x_d) = U(x_1) \) (as in C.-Owen).
Sufficient conditions for existence

- The asymptotic satiability of $u(\mathcal{E} + x)$ is difficult to check. Nonetheless ...
- Let $U^*$ satisfy the growth condition

$$U^*(\epsilon x^*) \leq \zeta(\epsilon)(U^*(x^*)^+ + 1)$$

for all $x^* \in \mathbb{R}^d_+, \epsilon \in (0, 1]$ and for some positive function $\zeta$. Then, both $U$ and $x \mapsto u(\mathcal{E} + x)$ are asymptotically satiable if $\mathcal{E} \in \mathcal{O} \cap L^\infty$. 

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- Liquidation to, e.g., the first asset can be included in the picture taking $U(x_1 \ldots x_d) = U(x_1)$ (as in C.-Owen).
Let $B$ be a given contingent claim and let $j = 1, \ldots, d$.

**Definition**

The *utility indifference (bid) price* (UIP), expressed in units of asset $j$, for $B$ is the solution $p_j(B) = p_j(B; U, \mathcal{E}) \in \mathbb{R}$ to the equation

$$u(\mathcal{E} + B - e_jp_j) = u(\mathcal{E})$$

(4.1)

Define $m_j(X) := \frac{m(X)}{m_j(\Omega)}$ and $\hat{m}_j(X) := \inf_{m \in \mathcal{D} \cap \text{dom}(U^*_0)} m_j(X)$.

**Proposition**

Assume $\mathcal{E} \in \mathcal{O}$ and $\mathcal{E} + B - e_j\hat{m}_j(B) \in \mathcal{O}$. Thus there exists a unique UIP for $B$. 
Utility indifference pricing II

**Proposition**

Under the assumptions $E \in \mathcal{O}$ and $E + B - e_j \hat{m}_j(B) \in \mathcal{O}$ we have

1. $m_j(B) \leq p_j(B) \leq \hat{m}_j(B)$;
2. for $c \in \mathbb{R}$ we have $p_j(B + e_j c) = p_j(B) + c$;
3. if $B \preceq C$ then $p_j(B) \leq p_j(C)$ for any $j = 1, \ldots, d$;
4. given contingent claims $B_1, B_2$ and $\lambda \in [0, 1]$

$$p_j(\lambda B_1 + (1 - \lambda) B_2) \geq \lambda p_j(B_1) + (1 - \lambda) p_j(B_2)$$

for any $j = 1, \ldots, d$;
Utility indifference pricing III: dual representation

Proposition

Under the previous assumptions, the UIP can be expressed as

\[ p_j(B) = \inf_{m \in D^j(1) \cap \text{dom}(U^*_0)} \{ m_j(B) + \alpha_j(m) \} \]

where

\[ D^j(k) := \{ m \in D : m_j(1) = k \} \]

\[ \alpha_j(m) := \inf_{k > 0} \frac{1}{k} \left\{ \mathbb{E} \left[ U^* \left( \frac{dm^k,c}{dP} \right) \right] + m^k(\mathcal{E}) - v_{\mathcal{E}} \right\} \]

and \( m^k \) is such that \( m^k_i = m_i \) if \( i \neq j \) and \( m^k_j = km_j \).
Conclusions

- Utility function $U$ supported on $\mathbb{R}_+^d$ satisfying multivariate Inada's type conditions.
- No duality gap under the weak assumption $\mathcal{E} \in \mathcal{O}$.
- Existence of the optimizer under the stronger $\mathcal{E} \in \mathcal{O} \cap L^\infty$.
- Existence and uniqueness of UIP as in Owen-Zitkovic.
- UIP is convex risk measure, dual representation ...
- First step towards Kramkov-Sirbu type results for UIP’s asymptotic expansion in financial markets with proportional transaction costs ...

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