Weak Dynamic Programming
for Generalized State Constraints

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Outline

1. Handwaving Part
2. Rigorous Part
Outline

1 Handwaving Part

2 Rigorous Part
Controlled System

Ingredients:

- $\mathcal{U} =$ set of controls $\nu$
- Spacial position $x$ in the state space $S$ (separable metric)
- Time $t \in [0, T]$
- Controlled state process $X_{t,x}^\nu(\cdot)$ with values in $S$
  - $= \text{evolution under control } \nu \text{ for system started at } (t, x)$.

Example:

- $S = \mathbb{R}^d$, $\mathcal{U} =$ predictable processes, $X_{t,x}^\nu = \text{unique solution of}$
  
  $$X(s) = x + \int_t^s \mu(X(r), \nu_r) \, dr + \int_t^s \sigma(X(r), \nu_r) \, dW_r, \quad t \leq s \leq T,$$
Dynamic Programming Principle (DPP)

- Reward function $f : S \rightarrow \mathbb{R}$
- $\mathcal{U}(t, x) \subseteq \mathcal{U}$: controls admissible at $(t, x)$.
- Value function

$$V(t, x) := \sup_{\nu \in \mathcal{U}(t, x)} E[f(X^\nu_{t, x}(T))], \quad (t, x) \in [0, T] \times S$$

- Under suitable conditions, we expect DPP: for $s \in [t, T]$,

$$V(t, x) = \sup_{\nu \in \mathcal{U}(t, x)} E[V(s, X^\nu_{t, x}(s))].$$

“Optimization $t \rightsquigarrow T$ is equivalent to separate optimization $t \rightsquigarrow s$ and $s \rightsquigarrow T$.”

- Main use: derive equation for $V$ (e.g. Hamilton-Jacobi-Bellman PDE).
Recall DPP: \[ V(t, x) = \sup_{\nu \in \mathcal{U}(t, x)} E[V(s, X^\nu_{t,x}(s))] \]

\[ \leq \text{ “Continue using } \nu \text{ after } s”: \]

\[ E[f(X^\nu_{t,x}(T))] \leq E[V(s, X^\nu_{t,x}(s))] \quad \text{for all } \nu \in \mathcal{U}(t, x). \]

\[ \geq \text{ Given } \nu \in \mathcal{U}(t, x), \text{ construct } \tilde{\nu} \in \mathcal{U}(t, x) \text{ such that} \]

\[ E[f(X^{\tilde{\nu}}_{t,x}(T))] \geq E[V(s, X^\nu_{t,x}(s))] - \varepsilon. \]

- Recipe for \( \tilde{\nu} \): Use \( \nu \) up to time \( s \), then state is at \( y = X^\nu_{t,x}(s)(\omega) \).
  Now continue with \( \nu^* = \varepsilon \)-optimal control at \( (s, y) \).

Problems:
- \( V \) not known to be measurable: how to make sense of DPP?
- When \( S \) is uncountable: measurable selection problems since \( \nu^* \) depends on \( y \), hence on \( \omega \).
Recall DPP:  
\[
V(t, x) = \sup_{\nu \in \mathcal{U}(t, x)} E\left[V(s, X^\nu_{t,x}(s))\right]
\]

\(\leq\): "Continue using \(\nu\) after \(s\):"

\[
E[f(X^\nu_{t,x}(T))] \leq E[V(s, X^\nu_{t,x}(s))] \quad \text{for all } \nu \in \mathcal{U}(t, x).
\]

\(\geq\): Given \(\nu \in \mathcal{U}(t, x)\), construct \(\tilde{\nu} \in \mathcal{U}(t, x)\) such that

\[
E[f(X^{\tilde{\nu}}_{t,x}(T))] \geq E[V(s, X^\nu_{t,x}(s))] - \varepsilon.
\]

• Recipe for \(\tilde{\nu}\): Use \(\nu\) up to time \(s\), then state is at \(y = X^\nu_{t,x}(s)(\omega)\).
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Problems:
• \(V\) not known to be measurable: how to make sense of DPP?
• When \(S\) is uncountable: measurable selection problems since \(\nu^*\) depends on \(y\), hence on \(\omega\).
Construction of $\tilde{\nu}$ when $V$ is smooth

- Assume that $U(t,x) = U$.
- Selection made is easy if
  
  $$x \mapsto E[f(X_{t,x}^\nu(T))] \text{ lsc and } V \text{ usc} :$$

  Use same $\nu^*$ in a neighborhood of $y$ and exploit separability.

- Combine countably many $\varepsilon$-optimizers to construct $\tilde{\nu}$. 

Covering argument.
Weak DPP [Bouchard&Touzi 2011]

- $x \mapsto E[f(X_{t,x}^\nu(T))]$ Isc is usually okay, regularity for $V$ is a problem.
- Weak DPP: Replace $V$ by test function $\varphi$:

$$V(t, x) \geq \sup_{\nu \in \mathcal{U}} E[\varphi(s, X_{t,x}^\nu(s))] \text{ for all usc } \varphi \leq V.$$

- Covering argument works like for smooth $V$.
- For viscosity solutions, weak DPP is as good as DPP.
Add state constraint:

- Constraint that $X_{t,x}^\nu$ has to remain in given set $\mathcal{O} \subseteq \mathcal{S}$
- l.e., $\mathcal{U}(t, x) := \{ \nu \in \mathcal{U} : X_{t,x}^\nu(s) \in \mathcal{O} \text{ for } s \in [t, T], \text{ P-a.s.} \}$
- Under typical regularity:

$$
\nu^* \in \mathcal{U}(t, x) \not\Rightarrow \nu^* \in \mathcal{U}(t, x') \text{ even if } x' \text{ close to } x.
$$

Covering argument fails.
Deterministic case:

- Closed constraint

Stochastic case:

- Open constraint

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For $x'$ close to $x$, constraint $O$ will be violated with at most small probability $\delta \geq 0$.

Relaxed problem for fixed $\delta > 0$:

$$\mathcal{U}(t, x, \delta) := \{ \nu \in \mathcal{U} : P[X_{t,x}^\nu \in O] \geq 1 - \delta \}$$

- Corresponding value function $V(t, x, \delta)$.
- We expect $V(t, x, \delta) \downarrow V(t, x, 0) \equiv V(t, x)$ as $\delta \downarrow 0$. 
Idea for $V(t, x, \delta) \downarrow V(t, x, 0)$: manipulate $\nu \in \mathcal{U}(t, x, \delta)$

Switch to admissible control at distance $\varepsilon > 0$ to $\partial \mathcal{O}$. 
Expectation Constraints

General expectation constraint:

- Fix \( g : S \rightarrow \mathbb{R} \). For \( m \in \mathbb{R} \), define

\[
U(t, x, m) := \{ \nu \in U : E[g(X_{t,x}^\nu(T))] \leq m \},
\]

\[
V(t, x, m) := \sup_{\nu \in U(t,x,m)} E[f(X_{t,x}^\nu(T))].
\]

- Is of independent interest (quantile hedging, ...)

Example of state constraint:

- Augment \( X \) by \( Y_{t,x,y}^\nu(s) := y \land \inf_{r \in [t,s]} d_{\mathcal{O}^c}(X_{t,x}^\nu(r)) \)

- Take \( g(x, y) := 1_{(-\infty,0]}(y) \),

- then \( E[g(X_{t,x}^\nu(T), Y_{t,x,1}^\nu(T))] = P[X_{t,x}^\nu \text{ leaves } \mathcal{O}] \).

No dynamic programming for fixed \( m \).
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  $$\mathcal{U}(t, x, m) := \{ \nu \in \mathcal{U} : E[g(X_{t,x}^{\nu}(T))] \leq m \},$$

  $$V(t, x, m) := \sup_{\nu \in \mathcal{U}(t,x,m)} E[f(X_{t,x}^{\nu}(T))].$$

- Is of independent interest (quantile hedging, ...)

Example of state constraint:

- Augment $X$ by $Y_{t,x,y}^{\nu}(s) := y \wedge \inf_{r \in [t,s]} d_{\mathcal{O}^c}(X_{t,x}^{\nu}(r))$
- Take $g(x, y) := 1_{(-\infty,0]}(y)$,
- then $E[g(X_{t,x}^{\nu}(T), Y_{t,x,1}^{\nu}(T))] = P[X_{t,x}^{\nu} \text{ leaves } \mathcal{O}].$

No dynamic programming for fixed $m$. 

No switching for “leave $\mathcal{O}$ with probability $\leq 1/2$.”
Dynamic Formulation [Bouchard, Elie, Touzi 2009]

- Let $\mathcal{M}_{t,m}$ be a rich enough family of martingales $M$ with $E[M(\cdot)] = m$. Then

  \[\nu \in \mathcal{U}(t, x, m) \quad \iff \quad g(X_{t,x}^\nu(T)) \leq M(T) \text{ for some } M \in \mathcal{M}_{t,m}.\]

  \[\Rightarrow\text{ For } \nu \in \mathcal{U}(t, x, m), \text{ let } M(s) = E[g(X_{t,x}^\nu(T))|\mathcal{F}_s]. \text{ Then}\]

  \[g(X_{t,x}^\nu(T)) \leq M(T) \quad \text{and} \quad E[M(\cdot)] = E[g(X_{t,x}^\nu(T))] \leq m.\]

  \[\Leftarrow \text{ Conversely, if } M \text{ is a martingale with } E[M(\cdot)] = m, \text{ then}\]

  \[g(X_{t,x}^\nu(T)) \leq M(T) \quad \text{implies} \quad E[g(X_{t,x}^\nu(T))] \leq E[M(T)] = m.\]

- Dynamic programming works.
Summary of Ideas

- Using test function in DPP allows treatment like for smooth value function.
- Covering argument fails for state constraints . . .
- . . . but works if constraints are relaxed.
- Relaxed problem has dynamic programming if the martingale formulation is used.

→ General (relaxed) weak DPP for expectation constraints.
→ Recover classical state constraint by passing to limit and eliminating the extra dimension for the martingale.
Summary of Ideas

- Using test function in DPP allows treatment like for smooth value function.
- Covering argument fails for state constraints . . .
- . . . but works if constraints are relaxed.
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→ General (relaxed) weak DPP for expectation constraints.
→ Recover classical state constraint by passing to limit and eliminating the extra dimension for the martingale.
Weak DPP for Expectation Constraints

**Theorem** (under natural assumptions):

Let

- \((t, x, m) \in [0, T] \times S \times \mathbb{R},\)
- \(\nu \in \mathcal{U}(t, x, m),\)
- \(M \in \mathcal{M}_{t,m}\) such that \(M(T) \geq g(X_{t,x}^\nu(T)).\)
- \(t \leq s \leq T.\)

(i) Let \(\varphi \geq V\) be a test function. Then

\[
E[f(X_{t,x}^\nu(T))] \leq E[\varphi(s, X_{t,x}^\nu(s), M(s))].
\]

(ii) Assume that \(x \mapsto E[g(X_{t,x}^\nu(T))]\) is u.s.c. and let \(\varphi \leq V\) be a test function (l.s.c.). Then

\[
V(t, x, m + \delta) \geq E[\varphi(s, X_{t,x}^\nu(s), M(s))]
\]

for all \(\delta > 0.\)
Example: Controlled Diffusion

- Controls: $U$-valued predictable square-integrable processes.
- $X_{t,x}^\nu(\cdot) = \text{unique strong solution of}$

$$X(s) = x + \int_t^s \mu(X(r), \nu_r) \, dr + \int_t^s \sigma(X(r), \nu_r) \, dW_r, \quad t \leq s \leq T,$$

- where $\mu : \mathbb{R}^d \times U \to \mathbb{R}^d$ and $\sigma : \mathbb{R}^d \times U \to \mathbb{M}^d$ jointly Lipschitz.
- $f : S \to \mathbb{R}$ lsc, quadratic growth, $f^-$ subquadratic growth.
- $g : S \to \mathbb{R}$ usc, quadratic growth, $g^+$ subquadratic growth.
PDE for Expectation Constraint

**Theorem:** Assume that $V$ is locally bounded on $\text{int} \, D$, $D := \{ V > -\infty \}$.

(i) $V^*$ is a viscosity subsolution on $\overline{D} \setminus \{ t = T \}$ of

$$-\partial_t \varphi + H^*(\cdot, D \varphi, D^2 \varphi) \leq 0.$$ 

(ii) $V^*$ is a viscosity supersolution on $\text{int} D$ of

$$-\partial_t \varphi + H^*(\cdot, D \varphi, D^2 \varphi) \geq 0.$$ 

- $H(x, p, Q) := \inf_{(u, a) \in U \times \mathbb{R}^d} \left( - L^{u,a}(x, p, Q) \right)$,
- $L^{u,a}(x, p, Q) := \mu_{X,M}(x, u) \top p + \frac{1}{2} \text{Tr}[\sigma_{X,M}(x, u, a)Q]$,
- $\mu_{X,M}(x, u) := \begin{pmatrix} \mu(x, u) \\ 0 \end{pmatrix}$ and $\sigma_{X,M}(x, u, a) := \begin{pmatrix} \sigma(x, u) \\ a \top \mu(x, u) \end{pmatrix}$.
- $H^*(x, p, Q) := \limsup_{(x', p', Q') \to (x, p, Q)} H(x', p', Q')$,
- $V^*(t, x, m) := \limsup_{(t', x', m') \to (t, x, m) \in \text{int} D} V(t', x', m')$. 

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**Assumption C:** There exists a Lipschitz mapping \( \hat{u} : \mathcal{O} \to U \) such that for all \((t, x) \in [0, T] \times \mathcal{O}\), the process \( X = X_{t,x}^{\hat{u}(X)} \) stays in \( \mathcal{O} \), where

\[
X(s) = x + \int_t^s \mu(X(r), \hat{u}(X(r))) \, dr + \int_t^s \sigma(X(r), \hat{u}(X(r))) \, dW_r, \quad s \in [t, T].
\]

**Assumption D:** \( \mu(x, u), \sigma(x, u) \) have linear growth in \( x \), uniformly in \( u \).

→ Yields uniform integrability to get right-continuity of \( V \) as \( m \downarrow 0 \).
**Theorem:** Assume that $\bar{V}$ is locally bounded on $[0, T) \times \mathcal{O}$.

(i) $\bar{V}^*$ is a viscosity subsolution on $[0, T) \times \overline{\mathcal{O}}$ of

$$-\partial_t \varphi + \bar{H}^*(\cdot, D\varphi, D^2\varphi) \leq 0.$$ 

(ii) Under Assumptions C and D, $\bar{V}^*$ is a viscosity supersolution on $[0, T) \times \mathcal{O}$ of

$$-\partial_t \varphi + \bar{H}^*(\cdot, D\varphi, D^2\varphi) \geq 0.$$ 

- $\bar{H}(x, p, Q) := \inf_{u \in U} \left( - \bar{L}^u(x, p, Q) \right)$,
- $\bar{L}^u(x, p, Q) := \mu(x, u)^\top p + \frac{1}{2} \text{Tr} \left[ \sigma \sigma^\top (x, u) Q \right]$. 

$\bar{H}$ and $\bar{L}^u$ are defined as above.
Outline

1. Handwaving Part

2. Rigorous Part
Abstract Setup

- $(\Omega, \mathcal{F}, P)$ with filtration $\mathbb{F} = (\mathcal{F}_t)_{t \in [0, T]}, \ T \in (0, \infty)$
- Auxiliary filtrations $\mathbb{F}^t \subseteq \mathbb{F}$ for $t \in [0, T]$
- $S$ metric separable
- Augmented state spaces $S := [0, T] \times S$ and $\hat{S} := [0, T] \times S \times \mathbb{R}$
- $\mathcal{U}_t = \text{set of controls at time } t$

For $(t, x) \in S$:

- $X_{t,x}^\nu(\cdot)$ càdlàg $\mathbb{F}^t$-adapted $S$-valued process for $\nu \in \mathcal{U}_t$
- $f, g : S \rightarrow \mathbb{R}$ measurable functions such that $f(X_{t,x}^\nu(T)) \in L^1$, $g(X_{t,x}^\nu(T)) \in L^1$ for all $\nu \in \mathcal{U}_t$
- Define $F(t, x; \nu) := E[f(X_{t,x}^\nu(T))]$ and $G(t, x; \nu) := E[g(X_{t,x}^\nu(T))]$
- Admissible controls $\mathcal{U}(t, x, m) := \{\nu \in \mathcal{U}_t : G(t, x; \nu) \leq m\}$
- Value function $V(t, x, m) := \sup_{\nu \in \mathcal{U}(t, x, m)} F(t, x; \nu)$ for $(t, x, m) \in \hat{S}$
\( M_{t,0} = \) family of càdlàg martingales \( M = \{ M(s), s \in [t, T] \} \)

\( \mathbb{F}^t \)-adapted and with initial value \( M(t) = 0 \).

\( M_{t,m} := \{ m + M : M \in M_{t,0} \} \) for \( m \in \mathbb{R} \)

Richness: for all \((t, x) \in S\) and \( \nu \in \mathcal{U}_t \), there exists \( M^\nu_t[x] \in M_{t,m} \) such that \( M^\nu_t[x](T) = g(X^\nu_{t,x}(T)) \), where \( m = E[g(X^\nu_{t,x}(T))] \).

With \( M^+_{t,m,x}(\nu) := \{ M \in M_{t,m} : M(T) \geq g(X^\nu_{t,x}(T)) \} \), we have

\[ \mathcal{U}(t, x, m) = \{ \nu \in \mathcal{U}_t : M^+_{t,m,x}(\nu) \neq \emptyset \}. \]

\( \mathcal{T}^t = \) set of \( \mathbb{F}^t \)-stopping times with values in \([t, T] \)

**Assumption A:** For all \((t, x, m) \in \hat{S}, \nu \in \mathcal{U}(t, x, m), M \in M^+_{t,m,x}(\nu), \tau \in \mathcal{T}^t \) and \( P \)-a.e. \( \omega \in \Omega \), there exists \( \nu_\omega \in \mathcal{U}(\tau(\omega), X^\nu_{t,x}(\tau)(\omega), M(\tau)(\omega)) \) such that

\[ E[f(X^\nu_{t,x}(T))|\mathcal{F}_\tau](\omega) \leq F(\tau(\omega), X^\nu_{t,x}(\tau)(\omega); \nu_\omega). \]
\( \mathcal{M}_{t,0} \) = family of càdlàg martingales \( M = \{ M(s), s \in [t, T] \} \) \( \mathbb{F}^t \)-adapted and with initial value \( M(t) = 0 \).

\( \mathcal{M}_{t,m} := \{ m + M : M \in \mathcal{M}_{t,0} \} \) for \( m \in \mathbb{R} \)

**Richness:** for all \( (t, x) \in \mathcal{S} \) and \( \nu \in \mathcal{U}_t \), there exists \( M^\nu_t [x] \in \mathcal{M}_{t,m} \) such that \( M^\nu_t [x](T) = g(X^\nu_{t,x}(T)) \), where \( m = E[g(X^\nu_{t,x}(T))] \).

With \( \mathcal{M}^+_t, m, x(\nu) := \{ M \in \mathcal{M}_{t,m} : M(T) \geq g(X^\nu_{t,x}(T)) \} \), we have

\[ \mathcal{U}(t, x, m) = \{ \nu \in \mathcal{U}_t : \mathcal{M}^+_t, m, x(\nu) \neq \emptyset \}. \]

\( \mathcal{T}^t \) = set of \( \mathbb{F}^t \)-stopping times with values in \([t, T]\)

**Assumption A:** For all \( (t, x, m) \in \hat{\mathcal{S}} \), \( \nu \in \mathcal{U}(t, x, m) \), \( M \in \mathcal{M}^+_t, m, x(\nu) \), \( \tau \in \mathcal{T}^t \) and \( P \)-a.e. \( \omega \in \Omega \), there exists \( \nu_\omega \in \mathcal{U}(\tau(\omega), X^\nu_{t,x}(\tau)(\omega), M(\tau)(\omega)) \) such that

\[ E[f(X^\nu_{t,x}(T)) | \mathcal{F}_\tau](\omega) \leq F(\tau(\omega), X^\nu_{t,x}(\tau)(\omega); \nu_\omega). \]
Assumption B: Let \((t, x) \in S, \nu \in U_t, s \in [t, T], \bar{\nu} \in U_s\) and \(\Gamma \in \mathcal{F}_s^t\).

(B1) There exists a control \(\tilde{\nu} \in U_t\) such that

\[
X_{t,x}(\cdot) = X_{t,x}^\nu(\cdot) \quad \text{on } [t, T] \times (\Omega \setminus \Gamma);
\]

\[
X_{t,x}(\cdot) = X_{s,x}(s)(\cdot) \quad \text{on } [s, T] \times \Gamma;
\]

\[
E\left[ f\left( X_{t,x}^\nu(T) \right) \right] \geq F(s, X_{t,x}^\nu(s); \bar{\nu}) \quad \text{on } \Gamma.
\]

The control \(\tilde{\nu}\) is denoted by \(\nu \otimes_{(s, \Gamma)} \bar{\nu}\) and called a concatenation of \(\nu\) and \(\bar{\nu}\) on \((s, \Gamma)\).

(B2) Let \(M \in \mathcal{M}_{t,0}\). There exists a process \(\bar{M} = \{\bar{M}(r), r \in [s, T]\}\) such that

\[
\bar{M}(\cdot)(\omega) = \left( M_s^\nu [X_{t,x}^\nu(s)(\cdot)](\cdot) \right)(\omega) \quad \text{on } [s, T] \quad P\text{-a.s.}
\]

and

\[
M1_{[t,s]} + 1_{[s,T]} \left( M1_{\Omega \setminus \Gamma} + [\bar{M} - \bar{M}(s) + M(s)] 1_{\Gamma} \right) \in \mathcal{M}_{t,0}.
\]

(B3) Let \(m \in \mathbb{R}\) and \(M \in \mathcal{M}_{t,m,x}^+(\nu)\). For \(P\text{-a.e. } \omega \in \Omega\), there exist a control \(\nu_\omega \in U(s, X_{t,x}^\nu(s)(\omega), M(s)(\omega))\).
Example for concatenation: if controls are predictable processes,

\[ \nu \otimes_{(\tau, \Gamma)} \bar{\nu} := \nu 1_{[0, \tau]} + 1_{(\tau, \tau]}(\bar{\nu} 1_\Gamma + \nu 1_{\Omega \setminus \Gamma}) \]

Assumption B': Let \((t, x) \in S, \nu \in U_t, \tau \in T^t, \Gamma \in F^t_t\) and \(\bar{\nu} \in U_{\|\tau\|_L_\infty} \cdot\)

\((B0')\) \(U_s \supseteq U_{s'}\) for all \(0 \leq s \leq s' \leq T\).

\((B1')-(B3')\) Like (B1)–(B3), with \(s\) replaced by \(\tau\).

Domain is invariant:
Let \(D = \{(t, x, m) \in \hat{S} : U(t, x, m) \neq \emptyset\}\). Under (B3'),

\( (\tau, X_{t,x}^\nu(\tau), M(\tau)) \in D \)

for all \(\tau \in T^t, \nu \in U(t, x, m), M \in M_{t,m,x}^+(\nu)\).
**Theorem (DPP):**

Let \((t, x, m) \in \hat{S}, \nu \in \mathcal{U}(t, x, m), M \in \mathcal{M}_{t,m,x}^+(\nu)\) and \(\tau \in \mathcal{T}^t\) and let \(D \subseteq \hat{S}\) be a set such that \((\tau, X_{t,x}^\nu(\tau), M(\tau)) \in D\).

(i) Let Assumption A hold true and let \(\varphi : \hat{S} \to [-\infty, \infty]\) be a measurable function such that \(V \leq \varphi\) on \(D\). Then

\[
F(t, x; \nu) \leq E[\varphi(\tau, X_{t,x}^\nu(\tau), M(\tau))]. \tag{1}
\]

(ii) Let \(\delta > 0\), let Assumption B hold true and assume that \(\tau\) takes countably many values \((t_i)_{i \geq 1}\). Let \(\varphi : \hat{S} \to [-\infty, \infty]\) be a measurable function such that \(V \geq \varphi\) on \(D\). Assume that for fixed \(\bar{\nu} \in \mathcal{U}_{t_i}, \varphi(t_i, \cdot) \in USC, \ F(t_i, \cdot; \bar{\nu}) \in LSC, \ G(t_i, \cdot; \bar{\nu}) \in USC\). Then

\[
V(t, x, m + \delta) \geq E[\varphi(\tau, X_{t,x}^\nu(\tau), M(\tau))]. \tag{2}
\]

(ii') Let \(\delta > 0\), let Assumption B' hold true, \(V \geq \varphi\) on \(D\). Assume that that for fixed \(\bar{\nu} \in \mathcal{U}_{t_0}, \ t_0 \in [t, T], \varphi(\cdot) \in USC, \ F(\cdot; \bar{\nu}) \in LSC, \ G(\cdot; \bar{\nu}) \in USC\). Moreover, assume that \(\mathcal{D} \cap \mathcal{D}\) is \(\sigma\)-compact. Then (2) holds true.
Theorem (DPP):

Let \((t, x, m) \in \hat{S}, \nu \in \mathcal{U}(t, x, m), M \in \mathcal{M}^+_{t,m,x}(\nu)\) and \(\tau \in \mathcal{T}^t\) and let \(D \subseteq \hat{S}\) be a set such that \((\tau, X_{t,x}^\nu(\tau), M(\tau)) \in D\).

(i) Let Assumption A hold true and let \(\varphi : \hat{S} \to [-\infty, \infty]\) be a measurable function such that \(V \leq \varphi\) on \(D\). Then

\[
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\]  

(1)

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\[
V(t, x, m + \delta) \geq E[\varphi(\tau, X_{t,x}^\nu(\tau), M(\tau))].
\]  

(2)

(iii') Let \(\delta > 0\), let Assumption B' hold true, \(V \geq \varphi\) on \(D\). Assume that for fixed \(\bar{\nu} \in \mathcal{U}_{t_0}, t_0 \in [t, T], \varphi(\cdot) \in USC, F(\cdot; \bar{\nu}) \in LSC, G(\cdot; \bar{\nu}) \in USC\).
Moreover, assume that \(D \cap D\) is \(\sigma\)-compact. Then (2) holds true.
Theorem (DPP):

Let \((t, x, m) \in \hat{S}, \nu \in U(t, x, m), M \in \mathcal{M}_{t,m,x}^+(\nu)\) and \(\tau \in \mathcal{T}^t\) and let \(D \subseteq \hat{S}\) be a set such that \((\tau, X_{t,x}^\nu(\tau), M(\tau)) \in D\).

(i) Let Assumption A hold true and let \(\varphi : \hat{S} \to [-\infty, \infty]\) be a measurable function such that \(V \leq \varphi\) on \(D\). Then

\[
F(t, x; \nu) \leq E[\varphi(\tau, X_{t,x}^\nu(\tau), M(\tau))].
\] (1)

(ii) Let \(\delta > 0\), let Assumption B hold true and assume that \(\tau\) takes countably many values \((t_i)_{i \geq 1}\). Let \(\varphi : \hat{S} \to [-\infty, \infty]\) be a measurable function such that \(V \geq \varphi\) on \(D\). Assume that for fixed \(\bar{\nu} \in U_{t_i}, \varphi(t_i, \cdot) \in USC, \quad F(t_i, \cdot; \bar{\nu}) \in LSC, \quad G(t_i, \cdot; \bar{\nu}) \in USC\). Then

\[
V(t, x, m + \delta) \geq E[\varphi(\tau, X_{t,x}^\nu(\tau), M(\tau))].
\] (2)

(ii') Let \(\delta > 0\), let Assumption B' hold true, \(V \geq \varphi\) on \(D\). Assume that that for fixed \(\bar{\nu} \in U_{t_0}, \quad t_0 \in [t, T], \quad \varphi(\cdot) \in USC, \quad F(\cdot; \bar{\nu}) \in LSC, \quad G(\cdot; \bar{\nu}) \in USC\).

Moreover, assume that \(D \cap D\) is \(\sigma\)-compact. Then (2) holds true.
Application to State Constraints

Setup:

- \( \mathcal{O} \subseteq S := \mathbb{R}^d \) open.
- \( \bar{\mathcal{U}}(t, x) = \{ \nu \in \mathcal{U}_t : X^\nu_{t,x}(s) \in \mathcal{O} \text{ for all } s \in [t, T], P\text{-a.s.} \} \).
- \( \bar{V}(t, x) = \sup_{\nu \in \bar{\mathcal{U}}(t, x)} F(t, x; \nu) \text{ for } (t, x) \in S \)
- \( X^\nu_{t,x} \) has continuous paths.
- \( (t, x) \mapsto X^\nu_{t,x}(\cdot) \) is continuous in probability, uniformly in time.
- \( \bar{\mathcal{U}}(t, x) \neq \emptyset \) for \( (t, x) \in [0, T] \times \mathcal{O} \).

Assumption \( \bar{A} \): For all \( (t, x) \in S, \nu \in \bar{\mathcal{U}}(t, x), \tau \in \mathcal{T}^t \) and \( P\text{-a.e. } \omega \in \Omega \), there exists \( \nu_\omega \in \bar{\mathcal{U}}(\tau(\omega), X^\nu_{t,x}(\tau)(\omega)) \) such that

\[
E \left[ f(X^\nu_{t,x}(T)) \big| \mathcal{F}_\tau \right](\omega) \leq F(\tau(\omega), X^\nu_{t,x}(\tau)(\omega); \nu_\omega).
\]

Marcel Nutz (ETH)
Weak Dynamic Programming
**Application to State Constraints**

**Setup:**

- $\mathcal{O} \subseteq S := \mathbb{R}^d$ open.
- $\bar{U}(t, x) = \{ \nu \in \mathcal{U}_t : X^\nu_{t,x}(s) \in \mathcal{O} \text{ for all } s \in [t, T], \text{ } P\text{-a.s.} \}$.  
- $\bar{V}(t, x) = \sup_{\nu \in \bar{U}(t, x)} F(t, x; \nu)$ for $(t, x) \in S$
- $X^\nu_{t,x}$ has continuous paths.
- $(t, x) \mapsto X^\nu_{t,x}(\cdot)$ is continuous in probability, uniformly in time.
- $\bar{U}(t, x) \neq \emptyset$ for $(t, x) \in [0, T] \times \mathcal{O}$.

**Assumption $\bar{A}$:** For all $(t, x) \in S$, $\nu \in \bar{U}(t, x)$, $\tau \in \mathcal{T}^t$ and $P$-a.e. $\omega \in \Omega$, there exists $\nu_\omega \in \bar{U}(\tau(\omega), X^\nu_{t,x}(\tau)(\omega))$ such that

\[
E \left[ f(X^\nu_{t,x}(T)) \big| \mathcal{F}_\tau \right](\omega) \leq F(\tau(\omega), X^\nu_{t,x}(\tau)(\omega); \nu_\omega).
\]
Application to State Constraints

Special case of expectation constraint:
- $Y_{t,x,y}^\nu (s) := y \land \inf_{r \in [t,s]} d_{\mathcal{O}^c} (X_{t,x}^\nu (r))$
- Consider $\tilde{X}_{t,x,y}^\nu := (X_{t,x}^\nu, Y_{t,x,y}^\nu)$ on $\mathbb{R}^d \times \mathbb{R}$.

Recall: $E[g(\tilde{X}_{t,x}^\nu (T))] = P[X_{t,x}^\nu \text{ leaves } \mathcal{O}]$ for $g(x,y) := 1_{(-\infty,0]}(y)$
DPP for State Constraints

**Theorem:** Consider \((t, x) \in S\) and a family \(\{\tau^\nu, \nu \in \bar{U}(t, x)\} \subseteq T^t\).

(i) Let Assumption \(\bar{A}\) hold true and let \(\phi\) be a measurable function such that \(\bar{V} \leq \phi\). Then

\[
\bar{V}(t, x) \leq \sup_{\nu \in \bar{U}(t, x)} E \left[ \phi(\tau^\nu, X_{t, x}^\nu(\tau^\nu)) \right].
\]

(ii) Let Assumption \(B'\) hold true for the state process \(\bar{X}\) on \(\mathbb{R}^d \times \mathbb{R}\).

Assume that

\[
V(t, x, 1, 0) = V(t, x, 1, 0+)
\]

and that \(F(\cdot; \nu) \in LSC([0, t_0] \times \mathcal{O})\) for all \(t_0 \in [t, T]\) and \(\nu \in \mathcal{U}_{t_0}\). Then

\[
\bar{V}(t, x) \geq \sup_{\nu \in \bar{U}(t, x)} E \left[ \phi(\tau^\nu, X_{t, x}^\nu(\tau^\nu)) \right]
\]

for any \(\phi \in USC\) such that \(\bar{V} \geq \phi\).
Setup for Controlled Diffusion I

- \( S = \mathbb{R}^d, \quad \Omega = C([0, T]; \mathbb{R}^d) \), \( W = \) canonical process, \( P = \) Wiener measure
- \( F = \mathbb{F}^W \) and \( F_t = \) filtration generated by \( W_s - W_t \), \( s \geq t \).
- \( U = \) set of \( U \)-valued predictable \( \nu \) with \( E[\int_0^T |\nu_t|^2 \, dt] < \infty \), where \( U \subseteq \mathbb{R}^d \) closed subset.
- \( U_t = \{ \nu \in U : \nu \) is \( F_t \)-predictable\}

\[ X_{t,x}^\nu(\cdot) = \text{unique strong solution of} \]

\[ X(s) = x + \int_t^s \mu(X(r), \nu_r) \, dr + \int_t^s \sigma(X(r), \nu_r) \, dW_r, \quad t \leq s \leq T, \]

- where \( \mu : \mathbb{R}^d \times U \rightarrow \mathbb{R}^d \) and \( \sigma : \mathbb{R}^d \times U \rightarrow M^d \) jointly Lipschitz.
Setup for Controlled Diffusion II

- $\mathcal{M}_{t,0} = \text{càdlàg martingales starting at 0 and adapted to } \mathbb{F}^t$.
- Using Brownian representation: $\mathcal{M}_{t,0} = \{ \int_t^T \alpha_s^\top dW_s, \alpha \in \mathcal{A}_t \}$,
- $\mathcal{A}_t = \mathbb{F}^t$-predictable processes $\alpha$ such that $\int_0^T |\alpha_t|^2 \, dt < \infty$ $P$-a.s.

- $f : S \to \mathbb{R}$ lsc, quadratic growth, $f^-$ subquadratic growth.
- $g : S \to \mathbb{R}$ usc, quadratic growth, $g^+$ subquadratic growth.

Then the previous assumptions are satisfied.
PDE for Expectation Constraint

**Theorem:** Assume that $V$ is locally bounded on $\text{int} \ D$.

(i) $V^*$ is a viscosity subsolution on $\overline{D} \setminus \{t = T\}$ of

$$-\partial_t \varphi + H^*(\cdot, D\varphi, D^2\varphi) \leq 0.$$  

(ii) $V^*$ is a viscosity supersolution on $\text{int} \ D$ of

$$-\partial_t \varphi + H^*(\cdot, D\varphi, D^2\varphi) \geq 0.$$  

- $H(x, p, Q) := \inf_{(u,a) \in U \times \mathbb{R}^d} \left(- L^{u,a}(x, p, Q)\right)$,
- $L^{u,a}(x, p, Q) := \mu_{X,M}(x, u)^\top p + \frac{1}{2} \text{Tr}[\sigma_{X,M}(x, u, a) Q]$,
- $\mu_{X,M}(x, u) := \begin{pmatrix} \mu(x, u) \\ 0 \end{pmatrix}$ and $\sigma_{X,M}(x, u, a) := \begin{pmatrix} \sigma(x, u) \\ a^\top \end{pmatrix}$,
- $H^*(x, p, Q) := \limsup_{(x', p', Q') \to (x, p, Q)} H(x', p', Q')$,
- $V^*(t, x, m) := \limsup_{(t', x', m') \to (t, x, m)} V(t', x', m')$.
Assumption C: There exists a Lipschitz mapping \( \hat{u} : \mathcal{O} \to U \) such that for all \((t, x) \in [0, T] \times \mathcal{O},\) the process \( X = X_{t, x}^{\hat{u}}(X) \) stays in \( \mathcal{O}, \) where

\[
X(s) = x + \int_t^s \mu(X(r), \hat{u}(X(r))) \, dr + \int_t^s \sigma(X(r), \hat{u}(X(r))) \, dW_r, \quad s \in [t, T].
\]

Assumption D: \( \mu(x, u), \sigma(x, u) \) have linear growth in \( x, \) uniformly in \( u. \)

\( \rightarrow \) Yields uniform integrability to get right-continuity of \( V \) as \( m \downarrow 0. \)

\[
\begin{align*}
\bar{H}(x, p, Q) &:= \inf_{u \in U} \left( -\bar{L}^u(x, p, Q) \right), \\
\bar{L}^u(x, p, Q) &:= \mu(x, u)^\top p + \frac{1}{2} \text{Tr}[\sigma \sigma^\top(x, u)Q].
\end{align*}
\]
Theorem: Assume that $\bar{V}$ is locally bounded on $[0, T) \times \mathcal{O}$.

(i) $\bar{V}^*$ is a viscosity subsolution on $[0, T) \times \mathcal{O}$ of

$$-\partial_t \varphi + \bar{H}^*\left(\cdot, D\varphi, D^2\varphi\right) \leq 0.$$ 

(ii) Under Assumptions C and D, $\bar{V}^*$ is a viscosity supersolution on $[0, T) \times \mathcal{O}$ of

$$-\partial_t \varphi + \bar{H}^*\left(\cdot, D\varphi, D^2\varphi\right) \geq 0.$$
PDE for State Constraint: Uniqueness

**Proposition:** Let $f$ be continuous and let Assumptions C and D hold true. Then

$$\bar{V}^*(T, \cdot) \leq f \quad \text{and} \quad \bar{V}_*(T, \cdot) \geq f \quad \text{on} \quad \bar{O}.$$ 

Assume in addition that $\bar{V}_*$ is of class $\mathcal{R}(O)$. Then

- $\bar{V}$ is continuous on $[0, T] \times O$ and admits a continuous extension to $[0, T] \times \bar{O}$,
- $\bar{V}$ is the unique (discontinuous) viscosity solution of

$$-\partial_t \varphi + \bar{H}(\cdot, D\varphi, D^2\varphi) = 0, \quad \varphi(T, \cdot) = f$$

in the class of functions having polynomial growth and having a lower semicontinuous envelope of class $\mathcal{R}(O)$. 

Marcel Nutz (ETH)
Regularity: Class $\mathcal{R}(\mathcal{O})$

**Proposition:** Consider a set $\mathcal{O} \subseteq \mathbb{R}^d$ and a function $w : [0, T] \times \overline{\mathcal{O}} \to \mathbb{R}$. Then $w$ is of class $\mathcal{R}(\mathcal{O})$ if the following hold for any $(t, x) \in [0, T) \times \partial \mathcal{O}$:

- There exist $r > 0$, an open neighborhood $B$ of $x$ in $\mathbb{R}^d$ and a function $\ell : \mathbb{R}_+ \to \mathbb{R}^d$ such that
  \[
  \liminf_{\varepsilon \to 0} \varepsilon^{-1} |\ell(\varepsilon)| < \infty \quad \text{and} \quad y + \ell(\varepsilon) + o(\varepsilon) \in \mathcal{O} \quad \text{for all } y \in \overline{\mathcal{O}} \cap B \text{ and } \varepsilon \in (0, r).
  \]

- There exists a function $\lambda : \mathbb{R}_+ \to \mathbb{R}_+$ such that
  \[
  \lim_{\varepsilon \to 0} \lambda(\varepsilon) = 0 \quad \text{and} \quad \lim_{\varepsilon \to 0} w(t + \lambda(\varepsilon), x + \ell(\varepsilon)) = w(t, x).
  \]
Sufficient Condition for $\mathcal{R}(\mathcal{O})$

**Proposition:** Let $\tilde{V}_*$ be finite-valued and assume:

- There exists a $C^1$-function $\delta$, defined on a neighborhood of $\overline{\mathcal{O}} \subseteq \mathbb{R}^d$, such that $D\delta$ is locally Lipschitz continuous and
  \[
  \delta > 0 \text{ on } \mathcal{O}, \quad \delta = 0 \text{ on } \partial \mathcal{O}, \quad \delta < 0 \text{ outside } \overline{\mathcal{O}}.
  \]

- There exists a locally Lipschitz continuous mapping $\tilde{u}: \mathbb{R}^d \to U$ such that for all $x \in \overline{\mathcal{O}}$ there exist an open neighborhood $B$ of $x$ and $\iota > 0$ satisfying
  \[
  \mu(z, \tilde{u}(z))^\top D\delta(y) \geq \iota \quad \text{and} \quad \sigma(y, \tilde{u}(y)) = 0 \quad \text{for all } y \in B \cap \overline{\mathcal{O}} \text{ and } z \in B.
  \]

Then $\tilde{V}_*$ is of class $\mathcal{R}(\mathcal{O})$. 

On Closed State Constraints

\[ \overline{V}(t, x) := \sup \left\{ E[f(X_{t,x}^\nu(T))] : \nu \in \mathcal{U}_t, X_{t,x}^\nu(\cdot) \in \overline{O} \right\}. \]

**Proposition:** Let \( f \) be continuous, let Assumptions C (on \( \overline{O} \)) and D hold true and assume that \( \overline{V}_* \) is of class \( \mathcal{R}(O) \). Then \( \overline{V} = \overline{V} \) on \([0, T] \times O\).

- One side of DPP works \( \Rightarrow \overline{V}_* \) subsolution of PDE.
- Clearly \( \overline{V} \geq \overline{V} \).
- Use comparison result.
On Closed State Constraints

\[ \overline{V}(t, x) := \sup \{ E[f(X_{t,x}(T))] : \nu \in \mathcal{U}_t, X_{t,x}^\nu(\cdot) \in \overline{\mathcal{O}} \}. \]

**Proposition:** Let \( f \) be continuous, let Assumptions C (on \( \overline{\mathcal{O}} \)) and D hold true and assume that \( \overline{V}_* \) is of class \( \mathcal{R}(\mathcal{O}) \). Then \( \overline{V} = \overline{V} \) on \([0, T] \times \mathcal{O}\).

- One side of DPP works \( \Rightarrow \overline{V}_* \) subsolution of PDE.
- Clearly \( \overline{V} \geq \overline{V} \).
- Use comparison result.
Consider $\rho \varphi - \partial_t \varphi + \mathcal{H}(\cdot, D\varphi, D^2\varphi) = 0$.

Assumption E:

There exists $\alpha > 0$ such that

$$\liminf_{\eta \downarrow 0} (\mathcal{H}(y, p, Y^n) - \mathcal{H}(x, p + q, X^n))$$

$$\leq \alpha \left(|x - y| (1 + |p| + n^2 |x - y|) + (1 + |x|) |q| + (1 + |x|^2) |Q| \right)$$

for all $(x, y) \in \overline{O}$ with $|x - y| \leq 1$ and for all $(p, q, Q) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathcal{M}^2d$, $(X^n, Y^n)_{\eta > 0} \subset \mathcal{M}^d \times \mathcal{M}^d$ and $n \geq 1$ such that

$$\begin{pmatrix} X^n & 0 \\ 0 & -Y^n \end{pmatrix} \leq A_n + \eta A^2_n \quad \text{for all } \eta > 0,$$

where

$$A_n := n^2 \begin{pmatrix} I_d & -I_d \\ -I_d & I_d \end{pmatrix} + Q.$$
Theorem: Let Assumption E hold true,
- $w_1$ usc subsolution on $\overline{O}$,
- $w_2$ lsc supersolution on $O$.

If $w_1$ and $w_2$ have polynomial growth on $\overline{O}$ and if $w_2$ is of class $\mathcal{R}(O)$, then

$$w_2 \geq w_1 \text{ on } \{T\} \times \overline{O} \quad \text{implies} \quad w_2 \geq w_1 \text{ on } [0, T] \times \overline{O}.$$
Thanks for your attention!