A BSDE Approach to Stochastic Differential Games with Incomplete Information

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Overview

Description of the Game

Existence and Uniqueness of the Value Function

BSDE Characterization of the Value Function

Outlook
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General Setup (Aumann, R.J., Maschler, M. (1967)):

- Two opponent players: Player 1 minimizes a certain quantity, Player 2 maximizes.

- Both players observe the actions of the other one.

- \( i \in \{1, \ldots, I\} \) different scenarios. Before the game starts one is picked with probability \((p_i)_{i \in \{1, \ldots, I\}} \in \Delta(I)\).

- Player 1 knows in which scenario the game is played.

- Player 2 only knows \((p_i)_{i \in \{1, \ldots, I\}}\).
Underlying Dynamics

Let \((\Omega, \mathcal{F}, (\mathcal{F}_s)_{s \in [0,T]}, \mathbb{P})\) be a filtered probability space with the usual assumptions, which carries a \(d\)-dimensional Brownian motion \((B_s)_{s \in [0,T]}\).

For all \(t \in [0, T], x \in \mathbb{R}^d\) we consider the controlled diffusion

\[
\begin{align*}
    dX^{t,x,u,v}_s &= b(s, X^{t,x,u,v}_s, u_s, v_s) \, ds + \sigma(s, X^{t,x,u,v}_s) \, dB_s \\
    X^{t,x,u,v}_t &= x,
\end{align*}
\]

where \((u_s)_{s \in [t,T]}\) control of Player 1, \((v_s)_{s \in [t,T]}\) control of Player 2.

Assumption: controls only take their values in \(U, V\), compact subsets of some finite dimensional space.
Target

For $i = 1, \ldots, I$ we consider

(i) running costs: $l_i : [0, T] \times \mathbb{R}^d \times U \times V \to \mathbb{R}$

(ii) terminal payoffs: $g_i : \mathbb{R}^d \to \mathbb{R}$

and denote

$$J_i(t, x, u, v) = \mathbb{E} \left[ \int_t^T l_i(s, X_s^{t,x,u,v}, u_s, v_s) ds + g_i(X_T^{t,x,u,v}) \right].$$

Before the game starts $i \in \{1, \ldots, I\}$ is chosen with $p \in \Delta(I)$.

Player 1 chooses his control to minimize, Player 2 chooses his control to maximize the expected payoff.

Assumption:

- Both players observe their opponents control.
- Player 1 knows scenario $i \in \{1, \ldots, I\}$.
- Player 2 just knows the probability $p$. 
Standing Assumptions (A)

(i) \( b : [0, T] \times \mathbb{R}^d \times U \times V \to \mathbb{R}^d, \sigma : [0, T] \times \mathbb{R}^d \to \mathbb{R}^{d \times d}, \)
\((l_i)_{i \in I} : [0, T] \times \mathbb{R}^d \times U \times V \to \mathbb{R}, (g_i)_{i \in I} : \mathbb{R}^d \to \mathbb{R}\) are bounded, continuous, Lipschitz w.r.t. \((t, x)\) uniformly in \((u, v)\).

(ii) For all \((t, x) \in [0, T] \times \mathbb{R}^d\) \(\sigma^T(t, x)\) is non-singular and \((\sigma^T)^{-1}(t, x)\) is bounded, Lipschitz.

(iii) Isaacs condition: for all \((t, x, \xi, p) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}^d \times \Delta(I)\)

\[
\inf_{u \in U} \sup_{v \in V} \left\{ \langle b(t, x, u, v), \xi \rangle + \sum_{i=1}^I p_i l_i(t, x, u, v) \right\} = \sup_{v \in V} \inf_{u \in U} \left\{ \langle b(t, x, u, v), \xi \rangle + \sum_{i=1}^I p_i l_i(t, x, u, v) \right\}
\]
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Strategies for Games with Information Incompleteness

Strategy for

(Players 1) \( \alpha : (v_s)_{s \in [t, t']} \rightarrow u_{t'} = \alpha (t', (B_s)_{s \in [t, t']}, (v_s)_{s \in [t, t']}) \)

(Players 2) \( \beta : (u_s)_{s \in [t, t']} \rightarrow v_{t'} = \beta (t', (B_s)_{s \in [t, t']}, (u_s)_{s \in [t, t']}) \).

Games with Incomplete Information

(i) Players learn new information during the game and adapt their behavior.

\[ J_i(t, x, \alpha, \beta) := J_i(t, x, u, v) \]

with \((u, v)\) s.t. \(u = \alpha(v), v = \beta(u)\).

(ii) Players try to hide their information in an optimal way.
Strategies for Games with Information Incompleteness

Random strategy for

(Player 1) \( \alpha_\omega : (v_s)_{s \in [t, t']} \rightarrow u_{t'} = \alpha_\omega(t', (B_s)_{s \in [t, t']}, (v_s)_{s \in [t, t']}) \)

(Player 2) \( \beta_\omega : (u_s)_{s \in [t, t']} \rightarrow v_{t'} = \beta_\omega(t', (B_s)_{s \in [t, t']}, (u_s)_{s \in [t, t']}) \).

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(i) Players learn new information during the game and adapt their behavior.

\( \rightarrow \) Play strategy vs. strategy

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\( \rightarrow \) They add randomness to their behavior.
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Games with Incomplete Information

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(ii) Players try to hide their information in an optimal way.

→ They add randomness to their behavior.

Denote \( A^r(t) \) (resp. \( B^r(t) \)) the set of random strategies for Player 1 (resp. 2), which are non anticipative with delay.
Value of the Game

For any \((t, x, p) \in [0, T] \times \mathbb{R}^d \times \Delta(l), \bar{\alpha} \in (\mathcal{A}^r(t))^l, \beta \in \mathcal{B}^r(t)\) set

\[
J(t, x, p, \bar{\alpha}, \beta) = \sum_{i=1}^l p_i J_i(t, x, \bar{\alpha}_i, \beta).
\]

**Theorem ([CardaliaguetRainer09])**

*For any \((t, x, p) \in [0, T] \times \mathbb{R}^d \times \Delta(l)\) the value of the game \(V(t, x, p)\) is given by*

\[
V(t, x, p) = \inf_{\bar{\alpha} \in (\mathcal{A}^r(t))^l} \sup_{\beta \in \mathcal{B}^r(t)} J(t, x, p, \bar{\alpha}, \beta)
= \sup_{\beta \in \mathcal{B}^r(t)} \inf_{\bar{\alpha} \in (\mathcal{A}^r(t))^l} J(t, x, p, \bar{\alpha}, \beta).
\] (1)
PDE Characterization

Define

\[ H(t, x, \xi, p) = \inf_{u \in U} \sup_{v \in V} \left\{ \langle b(t, x, u, v), \xi \rangle + \sum_{i=1}^{l} p_i l_i(t, x, u, v) \right\}. \]

Theorem ([Cardaliaguet09])

\[ V : [0, T[ \times \mathbb{R}^d \times \Delta(I) \to \mathbb{R} \] is the unique viscosity solution to

\[
\min \left\{ \frac{\partial w}{\partial t} + \frac{1}{2} \text{tr}(\sigma \sigma^T(t, x) D_x^2 w) + H(t, x, D_x w, p), \right. \\
\left. \lambda_{\min} \left( \frac{\partial^2 w}{\partial p^2} \right) \right\} = 0
\]

with boundary

\[ w(T, x, p) = \sum_i p_i g_i(x), \]

where for all \( A \in S^l \) \( \lambda_{\min}(A) \) denotes the smallest EV of \( A \).
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Canonical space

We will work on \((\Omega, \mathcal{F}) = (\Omega_B \times \Omega_p, \mathcal{F}^B \otimes \mathcal{F}^p)\), where

- \(\Omega_B := \mathcal{C}([0, T]; \mathbb{R}^d)\) set of continuous functions from \(\mathbb{R}\) to \(\mathbb{R}^d\), constant on \((-\infty, 0]\) and on \([T, +\infty)\).
  Set \(B_s(\omega_B) = \omega(s) \ \forall \omega_B \in \Omega_B\). Denote \((\mathcal{F}^B_s)_{s \in \mathbb{R}}\) filtration generated by \(s \mapsto B_s\).

- \(\Omega_p := \mathcal{D}([0, T]; \Delta(I))\) set of càdlàg functions from \(\mathbb{R}\) to \(\Delta(I)\), constant on \((-\infty, 0)\) and on \([T, +\infty)\).
  Set \(p_s(\omega_p) = \omega_p(s) \ \forall \omega_p \in \Omega_p\). Denote \((\mathcal{F}^p_s)_{s \in \mathbb{R}}\) filtration generated by \(s \mapsto p_s\).

For any measure \(\mathbb{P}\) on \(\Omega\), denote by \(\mathbb{E}_\mathbb{P}[\cdot]\) the expectation w.r.t. \(\mathbb{P}\).
Set of probability measures $\mathcal{P}(t, p)$

For all $p \in \Delta(I)$, $t \in [0, T]$ denote by $\mathcal{P}(t, p)$ the set of probability measures $\mathbb{P}$ on $\Omega$ such that, under $\mathbb{P}$

(i) $(B_s)_{s \in [0, T]}$ is a Brownian motion,

(ii) $p$ is a martingale, such that:

- $p_s = p \ \forall s < t$,
- $p_s \in \{e_i, i = 1, ..., l\} \ \forall s \geq T \ \mathbb{P}\text{-a.s.}$,
- $p_T$ is independent of $(B_s)_{s \in [-\infty, T]}$.

Furthermore denote $\mathcal{H}^2(\mathbb{P})$ the space of all predictable processes $\theta$, such that $\mathbb{E}_\mathbb{P} \left[ \int_0^T \theta_s^2 ds \right] < \infty$, $\mathcal{I}^2(\mathbb{P}) = \{ \int \theta dB : \theta \in \mathcal{H}^2(\mathbb{P}) \}$ and $\mathcal{M}_0^2(\mathbb{P})$ the space of square integrable martingales null at time zero.
BSDE Formulation

Fix \((t, x, p) \in [0, T] \times \mathbb{R}^d \times \Delta(I), \mathbb{P} \in \mathcal{P}(t, p)\). Define

\[
dX_{s}^{t,x} = \sigma(s, X_{s}^{t,x})dB_{s} \quad X_{t}^{t,x} = x
\]

and consider similar to [HamadèneLepeltier95]

\[
Y_{s}^{t,x,p} = \langle p_T, g(X_{T}^{t,x}) \rangle + \int_{s}^{T} H(r, X_{r}^{t,x}, Z_{r}^{t,x,p}, p_{r})dr
- \int_{s}^{T} \sigma(r, X_{r}^{t,x})Z_{r}^{t,x,p}dB_{r} - N_{T}^{t,x,p} + N_{s}^{t,x,p}, \tag{3}
\]

where \(N^{t,x,p} \in \mathcal{M}_0^2(\mathbb{P})\) is strongly orthogonal to \(\mathcal{I}^2(\mathbb{P})\).

(A) \Rightarrow BSDE (3) has a unique solution

\[
(Y^{t,x,p}, Z^{t,x,p}, N^{t,x,p}) \in \mathcal{H}^2(\mathbb{P}) \times \mathcal{H}^2(\mathbb{P}) \times \mathcal{M}_0^2(\mathbb{P}).
\]
Representation of the Value via BSDE

For any $\mathbb{P} \in \mathcal{P}(t, p)$

$$Y_{t-}^{t,x,\mathbb{P}} = \mathbb{E}_{\mathbb{P}} \left[ \int_t^T H(r, X_r^{t,x}, Z_r^{t,x,\mathbb{P}}, p_r) dr + \langle p_T, g(X_T^{t,x}) \rangle \right]|_{F_{t-}}.$$

**Theorem**

*For all $(t, x, p) \in [0, T] \times \mathbb{R}^d \times \Delta(I)$ the value of the game with incomplete information $V(t, x, p)$ is given by*

$$V(t, x, p) = \text{essinf}_{\mathbb{P} \in \mathcal{P}(t,p)} Y_{t-}^{t,x,\mathbb{P}}. \quad (4)$$

**Note:** Definition of $\mathcal{P}(t, p) \Rightarrow \exists$ probability measure $\mathbb{Q}$ on $F_{t-}$, such that $\forall \mathbb{P} \in \mathcal{P}(t, p)$ it holds $\mathbb{P}|_{F_{t-}} = \mathbb{Q}$, hence $\text{essinf}_{\mathbb{P} \in \mathcal{P}(t,p)} Y_{t-}^{t,x,\mathbb{P}}$ is $\mathbb{Q}$-a.s. defined.
Idea of proof

Define

\[ W(t, x, p) = \text{essinf}_{\mathbb{P} \in \mathcal{P}(t,p)} Y_t^{t,x,\mathbb{P}} \quad \mathbb{Q}\text{-a.s.} \]

Show

\begin{itemize}
  \item W(t, x, p) is deterministic ([BuckdahnLi08])
  \item W(t, x, p) is convex in p, uniformly Lipschitz in x, p, uniformly Hölder in t
  \item Dynamic Programming Principle
  \item W(t, x, p) is viscosity solution to
    \[
    \min \left\{ \frac{\partial w}{\partial t} + \frac{1}{2} \text{tr}(\sigma \sigma^T(t, x) D_x^2 w) + H(t, x, D_x w, p), \right. \\
    \left. \lambda_{\text{min}} \left( \frac{\partial^2 w}{\partial p^2} \right) \right\} = 0.
    \]
  \item V(t, x, p) is unique viscosity solution
    \[ \Rightarrow W(t, x, p) = V(t, x, p) \]
\end{itemize}
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(Open) Questions

- Is there a $\mathbb{P}^* \in \mathcal{P}(t, p)$ under which
  $$V(t, x, p) = Y_{t-}^{t, x, \mathbb{P}^*}$$

- Can one find a characterization of an optimal $\mathbb{P}^* \in \mathcal{P}(t, p)$?

- Can one use the BSDE to derive optimal strategies for the informed player?

- Regularity?

- Numerics?
Literature


Thank you!