Not for Publication Appendix for: ‘A Predictability Test for a Small Number of Nested Models’

by Eleonora Granziera, Kirstin Hubrich, Hyungsik Roger Moon

1 Invariance of the Ranking of the MSPE Differences w.r.t. the Clark and West Adjustment

In the following we will prove that the ranking of the MSPE differences is invariant to the introduction of the adjustment suggested by Clark and West. The required result is presented in Theorem 3. For notational simplicity we omit the time index t.

Frist, we consider a sequence of nested models as follows: for $m = 0, ..., M$,

$$\begin{aligned}
\text{Model } 0 \quad &: \quad y = \beta_0^{*\prime} x_0 + u_0 \\
\text{Model } 1 \quad &: \quad y = \beta_1^{*\prime} x_1 + u_1 = \beta_{10}^{*\prime} x_0 + \beta_{11}^{*\prime} x_{11} + u_1 \\
&\quad \vdots \\
\text{Model } m \quad &: \quad y = \beta_m^{*\prime} x_m + u_m = \beta_{m0}^{*\prime} x_0 + \beta_{m1}^{*\prime} x_{11} + \cdots + \beta_{mm}^{*\prime} x_{mm} + u_m
\end{aligned}$$

and

$$\beta_m^* = E\left(x_m x_m'\right)^{-1} E\left(x_m y\right).$$

Denote

$$y_m = P_{x_m} y = \beta_m^{*\prime} x_m.$$

For random variables Z_1 and Z_2, $\|Z_1\| = \left(E\left(Z_1^2\right)\right)^{1/2}$ signifies the L_2 norm and $\langle Z_1, Z_2 \rangle$ is the inner product defined by $E\left(Z_1 Z_2\right)$. Then, by definition

$$\langle y_m, u_m \rangle = 0.$$

Lemma 1 $P_{x_k} u_m = 0$ *for any* $k \leq m$.

Proof. Notice by definition,

$$P_{x_k} u_m = x_k' E\left(x_k x_k'\right)^{-1} E\left(x_k u_m\right).$$

The required result follows because x_k is a subcomponent of x_m and

$$E\left(x_m u_m\right) = E\left(x_m y\right) - E\left(x_m x_m'\right)\beta_m^* = 0. \blacksquare$$

Lemma 2 $\left\|u_m\right\|^2 \geq \left\|u_{m+1}\right\|^2.$

Proof. The required result follows since

$$\begin{aligned}
\left\|u_m\right\|^2 &= \left\|y - P_{x_m}y\right\|^2 \\
&= \left\|y_{m+1} - P_{x_m}y + u_{m+1}\right\|^2 \\
&= \left\|y_{m+1} - P_{x_m}y_{m+1} + u_{m+1}\right\|^2 \text{ since } P_{x_m}u_{m+1} = 0 \text{ by Lemma 1} \\
&= \left\|y_{m+1} - P_{x_m}y_{m+1}\right\|^2 + \left\|u_{m+1}\right\|^2 + 2\left\langle y_{m+1} - P_{x_m}y_{m+1}, u_{m+1}\right\rangle \\
&= \left\|y_{m+1} - P_{x_m}y_{m+1}^2\right\| + \left\|u_{m+1}\right\|^2 \text{ since } \left\langle y_{m+1}, u_{m+1}\right\rangle = 0,\ \left\langle P_{x_m}y_{m+1}, u_{m+1}\right\rangle = 0. \blacksquare
\end{aligned}$$

Lemma 3 $\left\|y_m\right\|^2 \leq \left\|y_{m+1}\right\|^2.$

Proof. By definition,

$$\left\|y\right\|^2 = \left\|y_m\right\|^2 + \left\|u_m\right\|^2 = \left\|y_{m+1}\right\|^2 + \left\|u_{m+1}\right\|^2.$$

The required result follows since $\left\|u_m\right\|^2 \geq \left\|u_{m+1}\right\|^2$ by Lemma 2 $\blacksquare$

Lemma 4 $\left\|y_m - y_k\right\|^2 = \left\|y_m\right\|^2 - \left\|y_k\right\|^2$ *for any* $k < m$.

Proof. By definition

$$\begin{aligned}
&\left\|y_m - y_k\right\|^2 = \left\|y_m - P_{x_k}y\right\|^2 = \left\|y_m - P_{x_k}y_m - P_{x_k}u_m\right\|^2 \\
=\ &\left\|y_m - P_{x_k}y_m\right\|^2 \text{ since } P_{x_k}u_m = 0 \\
=\ &\left\|y_m\right\|^2 - \left\|P_{x_k}y_m\right\|^2 = \left\|y_m\right\|^2 - \left\|P_{x_k}y\right\|^2 = \left\|y_m\right\|^2 - \left\|y_k\right\|^2. \blacksquare
\end{aligned}$$

Theorem 3. Define $\mu_m^{adj} = \left\|u_0\right\|^2 - \left\|u_m\right\|^2 + \left\|y_m - y_0\right\|^2.$ Then,

$$\mu_m^{adj} = 2\left\|u_0\right\|^2 - 2\left\|u_m\right\|^2.$$

Proof. The required result follows since

$$\begin{aligned}
\mu_m^{adj} &= \|u_0\|^2 - \|u_m\|^2 + \|y_m - y_0\|^2 \\
&= \|u_0\|^2 - \|u_m\|^2 + \|y_m\|^2 - \|y_0\|^2 \\
&= \|u_0\|^2 - \|u_m\|^2 + \|y\|^2 - \|u_m\|^2 - \|y\|^2 + \|u_0\|^2 \\
&= 2\|u_0\|^2 - 2\|u_m\|^2 .
\end{aligned}$$

2 Asymptotic Analysis

In this section we derive the asymptotic distribution of the test statistics $\mathcal{T}_{LRT}$. We follow closely Clark and McCracken (2001a, 2005, 2010) for the basic approximation results. Here we provide only a sketch of the proofs and cite Clark and McCracken (2001b) for more detailed derivations.

Lemma 5 *Assume Assumptions 1 – 4. Then, for all* $m = 1, ..., M$,

$$\frac{1}{2}(P-\tau+1)\bar{f}_m^{adj} = \sum_{t=T}^{T+P-\tau} \frac{1}{\frac{t}{T}} \left(\frac{1}{\sqrt{T}} \sum_{s=1}^{t-\tau} \tilde{h}_s \right)' Q_m \frac{\tilde{h}_{t+\tau}}{\sqrt{T}} + o_p(1) .$$

Proof. Notice by definition that

$$\frac{1}{2}(P-\tau+1)\bar{f}_m^{adj} = \sum_{t=T}^{T+P-\tau} \hat{u}_{0,t+\tau}\left(\hat{u}_{0,t+\tau} - \hat{u}_{m,t+\tau}\right).$$

Under the null hypothesis, we have $\hat{u}_{m,t+\tau} = u_{t+\tau} - x'_{m,t}\left(\hat{\beta}_{m,t} - \beta^0_m\right)$. By similar argument used in Lemma A10 of Clark and McCarcken (2001b), we can deduce the required result

$$\begin{aligned}
& \sum_{t=T}^{T+P-\tau} \hat{u}_{0,t+\tau}\left(\hat{u}_{0,t+\tau} - \hat{u}_{m,t+\tau}\right) \\
= \; & -\sum_{t=T}^{T+P-\tau} \left(\hat{\beta}_{0,t} - \beta^0_0\right)' x_{0,t} u_{t+\tau} + \sum_{t=T}^{T+P-\tau} \left(\hat{\beta}_{m,t} - \beta^0_m\right)' x_{m,t} u_{t+\tau} + o_p(1) \\
= \; & \sum_{t=T}^{T+P-\tau} \left(\frac{1}{t}\sum_{s=1}^{t-\tau} h_s\right)' \left[J'_m \left(J_m \left(\frac{1}{t}\sum_{s=1}^{t-\tau} x_s x'_s\right) J'_m\right)^{-1} J_m - J'_0 \left(J_0 \left(\frac{1}{t}\sum_{s=1}^{t-\tau} x_s x'_s\right) J'_0\right)^{-1} J_0\right] h_{t+\tau} \\
& + o_p(1). \\
= \; & \sum_{t=T}^{T+P-\tau} \frac{1}{\frac{t}{T}} \left(\frac{1}{\sqrt{T}}\sum_{s=1}^{t-\tau} h_s\right)' \left[J'_m \left(J_m \Sigma_x J'_m\right)^{-1} J_m - J'_0 \left(J_0 \Sigma_x J'_0\right)^{-1} J_0\right] \frac{h_{t+\tau}}{\sqrt{T}} + o_p(1) \\
= \; & \sum_{t=T}^{T+P-\tau} \frac{1}{\frac{t}{T}} \left(\frac{1}{\sqrt{T}}\sum_{s=1}^{t-\tau} \tilde{h}_s\right)' Q_m \frac{h_{t+\tau}}{\sqrt{T}} + o_p(1). \; \blacksquare
\end{aligned}$$

Lemma 6 *Assume Assumptions 1 – 4. Then, for all* $m = 1, ..., M$, *we have*

$$\begin{aligned}
& \frac{1}{4}(P-\tau+1)\hat{V} \\
= \; & \left[\frac{1}{T} \sum_{t:T \le t, t+j \le T+P-\tau} \frac{1}{\frac{t^2}{T^2}} \left\{ \left(\tfrac{1}{\sqrt{T}} \textstyle\sum_{s=1}^{t-\tau} \tilde{h}_s\right)' Q_m Q_n \left(\tfrac{1}{\sqrt{T}} \textstyle\sum_{s=1}^{t-\tau} \tilde{h}_s\right) \right\}\right]_{(m,n)} + o_p(1).
\end{aligned}$$

Proof. By definition and since $\bar{f}^{adj} = o_p(1)$, we have

$$\begin{aligned}
(P-\tau+1)\hat{V} &= \sum_{j=-\tau+1}^{\tau-1} \sum_{t:T \le t, t+j \le T+P-\tau} \left(\hat{f}^{adj}_{t+\tau} - \bar{f}^{adj}\right)\left(\hat{f}^{adj}_{t+\tau+j} - \bar{f}^{adj}\right)' \\
&= \sum_{j=-\tau+1}^{\tau-1} \sum_{t:T \le t, t+j \le T+P-\tau} \hat{f}^{adj}_{t+\tau}\left(\hat{f}^{adj}_{t+\tau+j}\right)' + o_p(1).
\end{aligned}$$

The $(m,n)^{th}$ element of $\sum_{j=-\tau+1}^{\tau-1}\sum_{t:T\leq t,t+j\leq T+P-\tau}\hat{f}^{adj}_{t+\tau}\left(\hat{f}^{adj}_{t+\tau+j}\right)'$ is

$$
\begin{aligned}
&\sum_{j=-\tau+1}^{\tau-1}\ \sum_{t:T\leq t,t+j\leq T+P-\tau}\hat{u}_{0,t+\tau}\hat{u}_{0,t+\tau+j}\left(\hat{u}_{0,t+\tau}-\hat{u}_{m,t+\tau}\right)\left(\hat{u}_{0,t+\tau+j}-\hat{u}_{n,t+\tau+j}\right)\\
=\ &\sum_{j=-\tau+1}^{\tau-1}\ \sum_{t:T\leq t,t+j\leq T+P-\tau}\left\{\begin{array}{c}\left(u_{t+\tau}-x_{0,t}'\left(\hat{\beta}_{0,t}-\beta_0^0\right)\right)\left(u_{t+\tau+j}-x_{0,t+j}'\left(\hat{\beta}_{0,t+j}-\beta_0^0\right)\right)\\ \times\left(x_{m,t}'\left(\hat{\beta}_{m,t}-\beta_m^0\right)-x_{0,t}'\left(\hat{\beta}_{0,t}-\beta_0^0\right)\right)\\ \times\left(x_{n,t+j}'\left(\hat{\beta}_{n,t+j}-\beta_n^0\right)-x_{0,t+j}'\left(\hat{\beta}_{0,t+j}-\beta_0^0\right)\right)\end{array}\right\}\\
=\ &\sum_{j=-\tau+1}^{\tau-1}\ \sum_{t:T\leq t,t+j\leq T+P-\tau}\left\{\begin{array}{c}\left(u_{t+\tau}x_{m,t}'\left(\hat{\beta}_{m,t}-\beta_m^0\right)-u_{t+\tau}x_{0,t}'\left(\hat{\beta}_{0,t}-\beta_0^0\right)\right)\\ \times\left(u_{t+\tau+j}x_{n,t+j}'\left(\hat{\beta}_{n,t+j}-\beta_n^0\right)-u_{t+\tau+j}x_{0,t+j}'\left(\hat{\beta}_{0,t+j}-\beta_0^0\right)\right)\end{array}\right\}+o_p(1)\\
=\ &\sum_{t:T\leq t,t+j\leq T+P-\tau}\left\{\begin{array}{c}\left(\frac{1}{t}\sum_{s=1}^{t-\tau}h_s\right)'\left[J_m'\left(J_m\Sigma_xJ_m'\right)^{-1}J_m-J_0'\left(J_0\Sigma_xJ_0'\right)^{-1}J_0\right]\\ \sum_{j=-\tau+1}^{\tau-1}\left(h_{t+\tau}h_{t+\tau+j}\right)\\ \left[J_n'\left(J_n\Sigma_xJ_n'\right)^{-1}J_n-J_0'\left(J_0\Sigma_xJ_0'\right)^{-1}J_0\right]\left(\frac{1}{t}\sum_{s=1}^{t-\tau+j}h_s\right)\end{array}\right\}+o_p(1)\\
=\ &\frac{1}{T}\sum_{t:T\leq t,t+j\leq T+P-\tau}\frac{1}{\frac{t^2}{T^2}}\left\{\begin{array}{c}\left(\frac{1}{\sqrt{T}}\sum_{s=1}^{t-\tau}h_s\right)'\left[J_m'\left(J_m\Sigma_xJ_m'\right)^{-1}J_m-J_0'\left(J_0\Sigma_xJ_0'\right)^{-1}J_0\right]\\ \times\sum_{j=-\tau+1}^{\tau-1}E\left(h_{t+\tau}h_{t+\tau+j}\right)\\ \left[J_n'\left(J_n\Sigma_xJ_n'\right)^{-1}J_n-J_0'\left(J_0\Sigma_xJ_0'\right)^{-1}J_0\right]\left(\frac{1}{\sqrt{T}}\sum_{s=1}^{t-\tau+j}h_s\right)\end{array}\right\}+o_p(1)\\
=\ &\frac{1}{T}\sum_{t:T\leq t,t+j\leq T+P-\tau}\frac{1}{\frac{t^2}{T^2}}\left\{\left(\frac{1}{\sqrt{T}}\sum_{s=1}^{t-\tau}\tilde{h}_s\right)'Q_mQ_n\left(\frac{1}{\sqrt{T}}\sum_{s=1}^{t-\tau}\tilde{h}_s\right)\right\}+o_p(1),
\end{aligned}
$$

as desired.

Lemma 7 *Assume Assumptions 1 – 4. Then*

$$
\begin{aligned}
&\left[\begin{array}{c}\frac{1}{\sqrt{T}}\sum_{s=1}^{[Tr]}\tilde{h}_s\\ \left(\begin{array}{c}\sum_{t=T}^{T+P-\tau}\frac{1}{\frac{t}{T}}\left(\frac{1}{\sqrt{T}}\sum_{s=1}^{t-\tau}\tilde{h}_s\right)'Q_1\frac{\tilde{h}_{t+\tau}}{\sqrt{T}}\\ \vdots\\ \sum_{t=T}^{T+P-\tau}\frac{1}{\frac{t}{T}}\left(\frac{1}{\sqrt{T}}\sum_{s=1}^{t-\tau}\tilde{h}_s\right)'Q_M\frac{\tilde{h}_{t+\tau}}{\sqrt{T}}\end{array}\right)\\ vech\left(\left[\frac{1}{T}\sum_{t:T\leq t,t+j\leq T+P-\tau}\frac{1}{\frac{t^2}{T^2}}\left\{\left(\frac{1}{\sqrt{T}}\sum_{s=1}^{t-\tau}\tilde{h}_s\right)'Q_mQ_n\left(\frac{1}{\sqrt{T}}\sum_{s=1}^{t-\tau}\tilde{h}_s\right)\right\}\right]_{(m,n)}\right)\end{array}\right]\\
\Rightarrow\ &\left[\begin{array}{c}W(r)\\ \left[\int_1^{1+\lambda}W(r)'Q_mdW(r)\right]_{(m)}\\ vech\left(\left[\int_1^{1+\lambda}W(r)'Q_mQ_nW(r)\right]_{(m,n)}\right)\end{array}\right]
\end{aligned}
$$

Proof. The required result follows by similar argument in the proof of Lemmas A5, A6, and A7 of Clark and McCracken (2001b).

Proof of Theorem 1

First, similar to Lemma A2 of Hansen (1990), under Assumption 3, we can show that

$$\mathcal{V} = \begin{bmatrix} \int_1^{1+\lambda} W(r)' Q_1 Q_1 W(r) & \cdots & \int_1^{1+\lambda} W(r)' Q_1 Q_M W(r) \\ \vdots & \ddots & \vdots \\ \int_1^{1+\lambda} W(r)' Q_M Q_1 W(r) & \cdots & \int_1^{1+\lambda} W(r)' Q_M Q_M W(r) \end{bmatrix} > 0 \text{ a.s.}$$

Also, the sets $\mathcal{A}_1, ..., \mathcal{A}_K$ are convex cones. Then, the required result follows by Lemmas 5, 6, and 7 and the continous mapping theorem. ■

The following notation will be used in the proof of Theroem 2. Given the simulate a sequence of *iid* $N(0,1)$ random variables $\eta_{s+\tau}$, define $u^*_{s+\tau} = \eta_{s+\tau}\varepsilon_{s+\tau} + \theta^0_1 \eta_{s+\tau-1}\varepsilon_{s+\tau-1} + \cdots + \theta^0_{\tau-1}\eta_{s+1}\varepsilon_{s+1}$. Recall the definition $\hat{u}^*_{s+\tau} = \eta_{s+\tau}\hat{\varepsilon}_{s+\tau} + \hat{\theta}_1 \eta_{s+\tau-1}\hat{\varepsilon}_{s+\tau-1} + \cdots + \hat{\theta}_{\tau-1}\eta_{s+1}\hat{\varepsilon}_{s+1}$. We denote $h^*_{s+\tau} = x_s u^*_{t+\tau}$, and $\hat{h}^*_{s+\tau} = x_s \hat{u}^*_{s+\tau}$.

Proof of Theorem 2

The proof is very similar to the proof of Theorems 3.3 and 3.4 of Clark and McCracken (2010), and we give only a sketch of the proof. Define $\hat{\beta}_m = \left(\hat{\beta}'_0, 0\right)'$. Then, $y^*_{t+\tau} = x'_{m,t}\hat{\beta}_m + \hat{u}^*_{t+\tau}$. Denote

$$\begin{aligned} \tilde{\beta}^*_{m,t} &= \left(\frac{1}{t}\sum_{s=1}^{t-\tau} x_{m,s}x'_{m,s}\right)^{-1}\left(\frac{1}{t}\sum_{s=1}^{t-\tau} x_{m,s}y^*_{t+\tau}\right) \\ &= \hat{\beta}_m + \left(J_m\left(\frac{1}{t}\sum_{s=1}^{t-\tau} x_s x'_s\right)J'_m\right)^{-1} J_m\left(\frac{1}{t}\sum_{s=1}^{t-\tau}\hat{h}^*_{s+\tau}\right) \end{aligned}$$

and

$$\begin{aligned} \tilde{u}^*_{t+\tau} &= y^*_{t+\tau} - x'_{m,t}\tilde{\beta}^*_{m,t} \\ &= \hat{u}^*_{t+\tau} - x'_{m,t}\left(\hat{\beta}_m - \tilde{\beta}^*_{m,t}\right) \\ &= \hat{u}^*_{t+\tau} - x'_t J'_m\left(J_m\left(\frac{1}{t}\sum_{s=1}^{t-\tau} x_s x'_s\right)J'_m\right)^{-1} J_m\left(\frac{1}{t}\sum_{s=1}^{t-\tau}\hat{h}^*_{s+\tau}\right) \end{aligned}$$

Notice that $\hat{f}^{*,adj}_{t+\tau} = \left(\hat{f}^{*,adj}_{1,t+\tau}, ..., \ \hat{f}^{*,adj}_{M,t+\tau}\right)'$, where

$$\hat{f}^{*,adj}_{m,t+\tau} = \tilde{u}^{*}_{0,t+\tau}\left(\tilde{u}^{*}_{0,t+\tau} - \tilde{u}^{*}_{m,t+\tau}\right).$$

Define $\bar{f}^{*,adj} = \frac{1}{P-\tau+1}\sum_{t=T}^{T+P-\tau}\left(\hat{f}^{*,adj}_{1,t+\tau}, ..., \hat{f}^{*,adj}_{M,t+\tau}\right)$. Then,

$$\hat{V}^{*} = \sum_{j=-\tau+1}^{\tau-1}\left[\frac{1}{P-\tau+1}\sum_{t:T\leq t,t+j\leq T+P-\tau}\left(\hat{f}^{*,adj}_{t+\tau} - \bar{f}^{*,adj}\right)\left(\hat{f}^{*,adj}_{t+\tau+j} - \bar{f}^{*,adj}\right)'\right]$$

and

$$\begin{aligned}\mathcal{T}^{*}_{LRT} &= (P-\tau+1)\max\Bigg\{\bar{f}^{*,adj\prime}\left(\hat{V}^{*}\right)^{-1}\bar{f}^{*,adj} - \min_{\mu\in\mathcal{A}_1}\left(\bar{f}^{*,adj}-\mu\right)'\left(\hat{V}^{*}\right)^{-1}\left(\bar{f}^{*,adj}-\mu\right), ...\\ &\quad ..., \bar{f}^{*,adj\prime}\left(\hat{V}^{*}\right)^{-1}\bar{f}^{*,adj} - \min_{\mu\in\mathcal{A}_K}\left(\bar{f}^{*,adj}-\mu\right)'\left(\hat{V}^{*}\right)^{-1}\left(\bar{f}^{*,adj}-\mu\right)\Bigg\}.\end{aligned}$$

First, following similar argument used in the proof of Theorems 3.3 and 3.4 of Clark and McCracken(2010), we can approximate that

$$\begin{aligned}&(P-\tau+1)\,\bar{f}^{*,adj}_{m}\\ =&\sum_{t=T}^{T+P-\tau}\tilde{u}^{*}_{0,t+\tau}\left(\tilde{u}^{*}_{0,t+\tau} - \tilde{u}^{*}_{m,t+\tau}\right)\\ =&\sum_{t=T}^{T+P-\tau}\left\{\hat{h}^{*\prime}_{t+\tau}\left\{\begin{array}{c} J'_m\left(J_m\left(\frac{1}{t}\sum_{s=1}^{t-\tau}x_s x'_s\right)J'_m\right)^{-1}J_m \\ -J'_0\left(J_0\left(\frac{1}{t}\sum_{s=1}^{t-\tau}x_s x'_s\right)J'_0\right)^{-1}J_0\end{array}\right\}\left(\frac{1}{t}\sum_{s=1}^{t-\tau}\hat{h}^{*}_{s+\tau}\right)\right\} + o^{*}_{p}(1)\\ =&\sum_{t=T}^{T+P-\tau}\left\{h^{*\prime}_{t+\tau}\left\{\begin{array}{c} J'_m\left(J_m\left(\frac{1}{t}\sum_{s=1}^{t-\tau}x_s x'_s\right)J'_m\right)^{-1}J_m \\ -J'_0\left(J_0\left(\frac{1}{t}\sum_{s=1}^{t-\tau}x_s x'_s\right)J'_0\right)^{-1}J_0\end{array}\right\}\left(\frac{1}{t}\sum_{s=1}^{t-\tau}\hat{h}^{*}_{s+\tau}\right)\right\} + o^{*}_{p}(1). \quad (1)\end{aligned}$$

Next, since $\bar{f}^{*,adj} = o^{*}_{p}(1)$, we have

$$(P-\tau+1)\,\hat{V}^{*} = \sum_{j=-\tau+1}^{\tau-1}\sum_{t:T\leq t,t+j\leq T+P-\tau}\hat{f}^{*,adj}_{t+\tau}\left(\hat{f}^{*,adj}_{t+\tau+j}\right)' + o^{*}_{p}(1).$$

Similarly, then, the $(m,n)^{th}$ element of $\sum_{j=-\tau+1}^{\tau-1}\sum_{t:T\leq t,t+j\leq T+P-\tau}\hat{f}_{t+\tau}^{*,adj}\left(\hat{f}_{t+\tau+j}^{*,adj}\right)'$ is approximated by

$$
\begin{aligned}
&\sum_{j=-\tau+1}^{\tau-1}\sum_{t:T\leq t,t+j\leq T+P-\tau}\tilde{u}_{0,t+\tau}^{*}\tilde{u}_{0,t+\tau+j}^{*}\left(\tilde{u}_{0,t+\tau}^{*}-\tilde{u}_{m,t+\tau}^{*}\right)\left(\tilde{u}_{0,t+\tau+j}^{*}-\tilde{u}_{n,t+\tau+j}^{*}\right)\\
= &\sum_{j=-\tau+1}^{\tau-1}\sum_{t:T\leq t,t+j\leq T+P-\tau}\left\{\begin{array}{c}\left(\hat{u}_{t+\tau}-x_{0,t}'\left(\tilde{\beta}_{0,t}^{*}-\hat{\beta}_{0}\right)\right)\left(\hat{u}_{t+\tau+j}-x_{0,t+j}'\left(\tilde{\beta}_{0,t+j}^{*}-\hat{\beta}_{0}\right)\right)\\ \times\left(x_{m,t}'\left(\tilde{\beta}_{m,t}^{*}-\hat{\beta}_{m}\right)-x_{0,t}'\left(\tilde{\beta}_{0,t}^{*}-\hat{\beta}_{0}\right)\right)\\ \times\left(x_{n,t+j}'\left(\tilde{\beta}_{n,t+j}^{*}-\hat{\beta}_{n}\right)-x_{0,t+j}'\left(\tilde{\beta}_{0,t+j}^{*}-\hat{\beta}_{0}\right)\right)\end{array}\right\}\\
= &\sum_{j=-\tau+1}^{\tau-1}\sum_{t:T\leq t,t+j\leq T+P-\tau}\left\{\begin{array}{c}\left(\hat{u}_{t+\tau}x_{m,t}'\left(\tilde{\beta}_{m,t}^{*}-\hat{\beta}_{m}\right)-\hat{u}_{t+\tau}x_{0,t}'\left(\tilde{\beta}_{0,t}^{*}-\hat{\beta}_{0}\right)\right)\\ \times\left(\hat{u}_{t+\tau+j}x_{n,t+j}'\left(\tilde{\beta}_{n,t+j}^{*}-\hat{\beta}_{n}\right)-\hat{u}_{t+\tau+j}x_{0,t+j}'\left(\tilde{\beta}_{0,t+j}^{*}-\hat{\beta}_{0}\right)\right)\end{array}\right\}+o_{p}^{*}(1)\\
= &\sum_{t:T\leq t,t+j\leq T+P-\tau}\left\{\begin{array}{c}\left(\frac{1}{t}\sum_{s=1}^{t-\tau}\hat{h}_{s}^{*}\right)'\left[\begin{array}{c}J_{m}'\left(J_{m}\left(\frac{1}{t}\sum_{s=1}^{t-\tau}x_{s}x_{s}'\right)J_{m}'\right)^{-1}J_{m}\\ -J_{0}'\left(J_{0}\left(\frac{1}{t}\sum_{s=1}^{t-\tau}x_{s}x_{s}'\right)J_{0}'\right)^{-1}J_{0}\end{array}\right]\\ \sum_{j=-\tau+1}^{\tau-1}\hat{h}_{t+\tau}^{*}\hat{h}_{t+\tau+j}^{*}\\ \left[\begin{array}{c}J_{n}'\left(J_{n}\left(\frac{1}{t}\sum_{s=1}^{t-\tau}x_{s}x_{s}'\right)J_{n}'\right)^{-1}J_{n}\\ -J_{0}'\left(J_{0}\left(\frac{1}{t}\sum_{s=1}^{t-\tau}x_{s}x_{s}'\right)J_{0}'\right)^{-1}J_{0}\end{array}\right]\left(\frac{1}{t}\sum_{s=1}^{t-\tau+j}\hat{h}_{s}^{*}\right)\end{array}\right\}+o_{p}^{*}(1)\\
= &\frac{1}{T}\sum_{t:T\leq t,t+j\leq T+P-\tau}\frac{1}{\frac{t^{2}}{T^{2}}}\left\{\begin{array}{c}\left(\frac{1}{\sqrt{T}}\sum_{s=1}^{t-\tau}h_{s}^{*}\right)'\left[\begin{array}{c}J_{m}'\left(J_{m}\left(\frac{1}{t}\sum_{s=1}^{t-\tau}x_{s}x_{s}'\right)J_{m}'\right)^{-1}J_{m}\\ -J_{0}'\left(J_{0}\left(\frac{1}{t}\sum_{s=1}^{t-\tau}x_{s}x_{s}'\right)J_{0}'\right)^{-1}J_{0}\end{array}\right]\\ \times\sum_{j=-\tau+1}^{\tau-1}h_{t+\tau}^{*}h_{t+\tau+j}^{*}\\ \left[\begin{array}{c}J_{m}'\left(J_{m}\left(\frac{1}{t}\sum_{s=1}^{t-\tau}x_{s}x_{s}'\right)J_{m}'\right)^{-1}J_{m}\\ -J_{0}'\left(J_{0}\left(\frac{1}{t}\sum_{s=1}^{t-\tau}x_{s}x_{s}'\right)J_{0}'\right)^{-1}J_{0}\end{array}\right]\left(\frac{1}{\sqrt{T}}\sum_{s=1}^{t-\tau+j}h_{s}^{*}\right)\end{array}\right\}+o_{p}^{*}(1).
\end{aligned}
$$

The required result follows by

$$\frac{1}{\sqrt{T}}\sum_{s=1}^{[Tr]}h_{s}^{*}\Rightarrow^{*}\Omega_{h}^{1/2}W(r)$$

(see Lemma 1 of Clark and McCracken (2010)) and the continuous mapping theorem. ■