Multi-Dimensional Integration Theorem

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1 Time Average Integration

Let $\bar{\mu}(t)$ represent a vector function of time taking values in $\mathbb{R}^N$. The sample average of $\bar{\mu}(t)$ taken at times $t_1, t_2, \ldots, t_m$ is written $\frac{1}{m} \sum_{i=1}^{m} \bar{\mu}(t_i)$. If $\bar{\mu}(t)$ takes values in a set $A$, then this average constitutes a convex combination of points in $A$, and hence is contained in the convex hull of $A$. Intuitively, the same result is true for time average integrals of $\bar{\mu}(t)$, because integrals can be represented as limits of finite sums. However, such a limiting argument cannot be used in general, as the set $A$ may not contain all its limit points. The following theorem proves the result by using the convex set separation theorem [1], which states that a convex set and a point not in the set can be separated by a hyperplane.

**Theorem 1.** (Time Average Integration) If $\bar{\mu}(t)$ is integrable and is contained within a set $A$ for all time, then the time average integral of $\bar{\mu}(t)$ over any finite interval of size $T$ is within the convex hull of $A$, i.e.:

$$\frac{1}{T} \int_0^T \bar{\mu}(t) dt \in \text{Conv}(A)$$

**Proof.** Suppose the result is true when the affine hull\(^1\) of $A$ has dimension less than or equal to $k-1$. The result is trivially true when $k-1 = 0$, as this implies $\bar{\mu}(t)$ is a single point for all time. We proceed by induction on $k$.

Assume the affine hull of $A$ has dimension $k$. By a simple change of coordinates, we can equivalently treat $\bar{\mu}(t)$ as a function taking values in $\mathbb{R}^k$. Let $\bar{\mu} = \frac{1}{T} \int_0^T \bar{\mu}(t) dt$. If the point $\bar{\mu}$ is within the set $\text{Conv}(A)$, we are done. If $\bar{\mu} \notin \text{Conv}(A)$, then by the convex set separation theorem there must exist a hyperplane $H$ which separates $\bar{\mu}$ from $\text{Conv}(A)$, i.e., there exists a vector $\bar{z}$ and a scalar $b$ such that

$$\bar{z}^T \bar{\mu} \leq b$$

$$\bar{z}^T \bar{a} \geq b \text{ for all } \bar{a} \in \text{Conv}(A)$$

\(^1\)The affine hull of a set $A$ is the set $\bar{a} + X$, where $\bar{a}$ is an arbitrary element of $A$, and $X$ is the smallest linear space such that $\bar{a} + X$ contains set $A$ [3]. For example, consider a set of points within $\mathbb{R}^N$ which all lie on the same plane, or the same line. Then the affine hull is the 2-dimensional plane, or, respectively, the 1-dimensional line.
where the hyperplane $H$ consists of all points $\tilde{x} \in \mathbb{R}^k$ such that $\tilde{x} \tilde{z} = b$. Thus, we have:

$$b \geq \tilde{z} \tilde{p} = \frac{1}{T} \int_0^T \tilde{z} \mu(t) dt$$  \hspace{1cm} (2)

However, $\tilde{\mu}(t) \in \text{Conv}(A)$ for all time, and hence by (1) the integrand in (2) is greater than or equal to $b$ for all time. This implies that the set of all times $t \in [0, T]$ for which $\tilde{z} \tilde{\mu}(t) > b$ must have measure zero. Hence:

$$\tilde{p} = \frac{1}{T} \int_0^T \tilde{\mu}(t) dt = \frac{1}{T} \int_{\{t \in [0, T] | \tilde{z} \tilde{\mu}(t) = b\}} \tilde{\mu}(t) dt$$  \hspace{1cm} (3)

The integral in (3) represents the time average of a function contained in the set $A \cap H$, a set of dimension at most $k - 1$. It follows by the induction hypothesis that $\tilde{p} \in \text{Conv}(A \cap H) \subset \text{Conv}(A)$, a contradiction.

\textbf{Corollary 1.} If the set $A$ is closed, then $\lim_{T \to \infty} \frac{1}{T} \int_0^T \tilde{\mu}(t) dt \in \text{Conv}(A)$, provided that the limit converges.

\textbf{Proof.} The limit can be approached arbitrarily closely by time average integrals over finite intervals. By Theorem 1, each such time average is contained within $\text{Conv}(A)$. The limiting integral is thus a limit point of the closed set $\text{Conv}(A)$, and hence is within $\text{Conv}(A)$.

Example: The corollary does not hold if the set $A$ is not closed. Indeed, consider the scalar valued function $\mu(t) = 1 - 1/(t + 1)$ contained within the non-closed interval $[0, 1)$ for all $t \geq 0$. Then the time average integral of $\mu(t)$ over any finite interval is within $[0, 1)$, but the limiting average as the interval size $T \to \infty$ is equal to 1, which is not in this interval.

\textbf{References}