Supplementary Notes — EE 549
Tuesday, April 12, 2005
I. Probabilistic Routing
II. M/G/1 Queues

I. PROBABILISTIC ROUTING

Consider a network of K nodes, and define:

- λ_i = Rate of exogenous traffic stream entering node i (for $i \in \{1, \dots, K\}$)
- P_{ij} = Probability of routing a packet leaving node i to a new node j
- γ_i = Aggregate rate into node i (from exogenous and endogenous streams)

We can write these quantities as row vectors $\vec{\lambda}$, $\vec{\gamma}$, and as a routing matrix $P = (P_{ij})$. Assuming the network is stable, the aggregate input rate γ_j into any node j is equal to the aggregate output rate from that node. It follows that for all nodes $j \in \{1, \dots, K\}$, we have:

$$\gamma_j = \lambda_j + \sum_i \gamma_i P_{ij}$$

or, written as a matrix:

$$\vec{\gamma} = \vec{\lambda} + \vec{\gamma} P$$

Hence, assuming stability, the aggregate rate vector is given by:

$$\vec{\gamma} = \vec{\lambda}(I - P)^{-1}$$

To ensure stability, we require that the service rate vector $\vec{\mu}$ is strictly larger than the aggregate input rate vector $\vec{\gamma}$, so that:

$$\vec{\lambda}(I - P)^{-1} < \vec{\mu}$$

This simple matrix inequality is all that is required to check stability for general networks with probabilistic routing. It shows which rate vectors $\vec{\lambda}$ can be supported in a given network, and likewise shows how to design the server rates $\vec{\mu}$ to handle a given set of traffic.

A. When is the $(I - P)$ matrix invertible?

The above manipulations implicitly assume that the $(I - P)$ matrix is invertible. It turns out that checking if this matrix is invertible requires only a simple thought experiment. Imagine an empty network, and inject a single packet at a particular source node. The packet bounces around the network according to the transition probability matrix P. Note that when the packet is in a certain node i, the probability it leaves the network is given by $1 - \sum_j P_{ij}$.[1]

The $(I - P)$ matrix is invertible whenever any such single packet eventually leaves the network (with probability 1), regardless of which source node it entered the network from. This is described in the following theorem.

Theorem 1: The $(I - P)$ matrix is invertible whenever $P^n \to (0)$, which occurs if and only if any single packet that enters the network will eventually leave. Furthermore, in such cases the inverse is equal to:

$$\begin{aligned}(I - P)^{-1} &= I + P + P^2 + P^3 + \dots \\ &= I + \sum_{n=1}^{\infty} P^n\end{aligned}$$

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[1] Note that, in general, the row sums of P are less than or equal to 1, but are strictly less than 1 for a particular row i in the case when a packet can exit the network from node i.

Proof: By a simple telescoping series argument we have the following identity:

$$(I-P)(I+P+P^2+\ldots+P^n) = I - P^{n+1} \tag{1}$$

It is not difficult to show that if $P^n \rightarrow 0$, then it must decrease geometrically, in which case the infinite sum $\sum_{n=1}^{\infty} P^n$ converges to a finite matrix. Thus, assuming that $P^n \rightarrow 0$, we take limits of (1) to obtain:

$$(I-P)\left(I + \sum_{n=1}^{\infty} P^n\right) = I$$

which implies that $(I-P)$ is invertible, and that $(I-P)^{-1} = (I + \sum_{n=1}^{\infty} P^n)$.

Now let $\vec{e}_i$ represent a unit row vector with a 1 in the i^{th} entry, and a 0 in every other entry. Considering the single packet thought experiment, it follows that if the packet is injected into the network at source node i, the probability the packet is in node j after n transitions is given by the j^{th} entry of the row vector $\vec{e}_i P^n$. But this converges to zero for every source node i if and only if P^n converges to the all-zero matrix as $n \rightarrow \infty$. □

II. M/G/1 QUEUES

An M/G/1 queue is a single-server queueing system with Poisson arrivals of jobs (with rate λ), where the service times of each job are independent and identically distributed according to a general distribution function $p_S(t)$. This is often called an 'M/GI/1' queue to emphasize the independence between service times of each job. Note that an M/M/1 queue with service rate μ can be described in this framework simply by defining $p_S(t) = \mu e^{-\mu t}$ (for $t \geq 0$).

Assume the system is in steady state, and consider a particular arrival from the Poisson stream. The time this job spends waiting in the buffer before beginning its own service time is given by:

$$W^q = R + \sum_{i=1}^{L^q} S_i$$

where the notation W^q emphasizes this is the time the job spends in the *queue* before beginning its service (this is different from the total system time W, equal to the sum of queueing time and service time). In the above equation, R is a random variable representing the amount of residual processing time required by the packet currently in the server, as seen by the new arrival ($R = 0$ if the new job arrives to an empty system). Likewise, L^q represents the number of jobs queued in the buffer and waiting for service at the time of arrival, and $S_1, \ldots, S_{L^q}$ are the service times of these jobs. Taking expectations, we thus have:

$$\begin{aligned} \mathbb{E}\{W^q\} &= \mathbb{E}\{R\} + \mathbb{E}\left\{\sum_{i=1}^{L^q} S_i\right\} \\ &= \mathbb{E}\{R\} + \mathbb{E}\{S\}\mathbb{E}\{L^q\} \end{aligned} \tag{2}$$

where $\mathbb{E}\{S\} = \int_0^{\infty} t p_S(t) dt$ is the expected service time of a job. The equation (2) follows by the fact that service times $\{S_i\}$ are independent of L^q, the current number of packets in queue as seen by the arrival. A more detailed derivation of (2) is given in Subsection II-A below.

By PASTA, the random variables W^q, R, and L^q as seen by the arriving packet are distributed according to their time average values. It follows that:

$$\begin{aligned} \mathbb{E}\{W^q\} &= \overline{W^q} \\ \mathbb{E}\{L^q\} &= \overline{L^q} \\ \mathbb{E}\{R\} &= \overline{R} \end{aligned}$$

where $\overline{R}$ is the time average residual time in the server, $\overline{L^q}$ is the time average number of packets in the queue, and $\overline{W^q}$ is the average waiting time of a packet in the queue. The equation (2) thus becomes:

$$\overline{W^q} = \overline{R} + \mathbb{E}\left\{S\right\} \overline{L^q} \tag{3}$$

However, by Little's Theorem, we know that:

$$\lambda \overline{W^q} = \overline{L^q} \tag{4}$$

Using (4) in (3) yields:

$$\overline{W^q} = \frac{\overline{R}}{1 - \lambda \mathbb{E}\left\{S\right\}} = \frac{\overline{R}}{1 - \rho}$$

where $\rho \triangleq \lambda \mathbb{E}\left\{S\right\}$ is the system loading. That is, ρ is the fraction of time the system is busy (which follows by Little's Theorem). The above equation establishes the expected waiting time in the queue for an M/G/1 system in terms of the average residual time $\mathbb{E}\left\{R\right\}$. To complete the derivation, we note that for any general queue with jobs arriving at rate λ, the time average residual time is given by $\overline{R} = \frac{\lambda \mathbb{E}\{S^2\}}{2}$ (see Subsection II-B below), so that the M/G/1 formula becomes:

$$\overline{W^q} = \frac{\lambda \mathbb{E}\left\{S^2\right\}}{2(1 - \rho)} \qquad \text{(P-K Formula)}$$

A. Deriving eq. (2)

Note that eq. (2) follows by the identity $\mathbb{E}\left\{\sum_{i=1}^{L^q} S_i\right\} = \mathbb{E}\left\{S\right\} \mathbb{E}\left\{L^q\right\}$. This can be proven by noting that the service times S_i are independent of the random variable L^q. We can thus use iterated expectations by conditioning on L^q. In particular, we have:

$$\begin{aligned}
\mathbb{E}\left\{\sum_{i=1}^{L^q} S_i\right\} &= \mathbb{E}\left\{\mathbb{E}\left\{\sum_{i=1}^{L^q} S_i \middle| L^q\right\}\right\} \\
&= \mathbb{E}\left\{\sum_{i=1}^{L^q} \mathbb{E}\left\{S_i \mid L^q\right\}\right\} \\
&= \mathbb{E}\left\{\sum_{i=1}^{L^q} \mathbb{E}\left\{S\right\}\right\} \\
&= \mathbb{E}\left\{S\right\} \mathbb{E}\left\{\sum_{i=1}^{L^q} 1\right\} = \mathbb{E}\left\{S\right\} \mathbb{E}\left\{L^q\right\} \quad \square
\end{aligned}$$

B. Computing the time average residual time

Let $\overline{R}$ represent the time average residual time of a packet in the server. That is, we define:

$$\overline{R} = \lim_{t \to \infty} \frac{1}{t} \int_0^t R(\tau) d\tau$$

where $R(\tau)$ is defined as the residual service time at the time instant τ (being zero if the system is empty at time τ). The integral of residual service time can be obtained by adding up the areas of all triangles defined by each individual job in the server. Let $N_s(t)$ represent the number of packets that have entered the server by time t. It follows that:

$$\begin{aligned}
\lim_{t \to \infty} \frac{1}{t} \int_0^t R(\tau) d\tau &= \lim_{t \to \infty} \frac{1}{t} \sum_{i=1}^{N_s(t)} \frac{S_i^2}{2} \\
&= \lim_{t \to \infty} \frac{N_s(t)}{t} \frac{1}{N_s(t)} \sum_{i=1}^{N_s(t)} \frac{S_i^2}{2} \\
&= \lambda \frac{\mathbb{E}\left\{S^2\right\}}{2}
\end{aligned}$$

where the last equality follows when $N_s(t) \to \infty$ (which is always true) and when $\frac{N_s(t)}{t} \to \lambda$ (which is true whenever the system is stable, i.e., when $\rho < 1$).