Supplementary Notes — EE 549 Spring 2008 Probabilistic Routing and Jackson Networks

I. Probabilistic Routing

Consider a network of K nodes. Packets arrive to each node from both exogenous arrival processes and endogenous processes consisting of departures from other nodes. The next node that a packet visits is chosen *probabilistically*, and there is the possibility of feedback to a previous node that has already been visited. Mathematically, we can describe the system with the following parameters:

- λ_i = Rate (in units of packets/sec) of exogenous traffic stream entering node i (for $i \in \{1, \ldots, K\}$)
- P_{ij} = Probability of routing a packet leaving node i to a new node j
- γ_i = Aggregate rate (packets/sec) into node i (from exogenous and endogenous streams)

We can write these quantities as row vectors $\vec{\lambda}$, $\vec{\gamma}$, and as a routing matrix $P = (P_{ij})$. Note that $1 - \sum_{j=1}^{N} P_{ij}$ is the *exit probability* associated with node i. That is, it is the probability that a packet does not visit any other node after node i, and hence exits the system. Assuming the network is stable, the aggregate input rate γ_j into any node j is equal to the aggregate output rate from that node. It follows that for all nodes $j \in \{1, \ldots, K\}$, we have:

$$\gamma_j = \lambda_j + \sum_i \gamma_i P_{ij}$$

The above equations must hold for all $j \in \{1, \ldots, N\}$. Thus, we have a system of N equations and N unknowns. These equations can be collectively written as a matrix as follows:

$$\vec{\gamma} = \vec{\lambda} + \vec{\gamma} P$$

Hence, assuming stability, the aggregate rate vector is given by:

$$\vec{\gamma} = \vec{\lambda}(I - P)^{-1} \tag{1}$$

Conditions that ensure the $(I - P)$ matrix are invertible are discussed in Appendix A.

II. Jackson Networks

Now let us assume the above network is a Jackson network, so that we have:

- Exogenous arrival processes are independent and Poisson with rate λ_i (for $i \in \{1, \ldots, N\}$)

- The Kleinrock Independence Assumption is used, so that each service time in node i is i.i.d. and exponential with rate μ_i (for $i \in \{1, \ldots, N\}$), regardless of the previous service times of the same packet in that node or in other nodes.

To ensure stability as well as the existence of a steady state distribution in this Jackson Network model, we require that the aggregate packet arrival rate γ_i into each node i (computed in (1)) is strictly less than μ_i. That is, we require $\gamma_i < \mu_i$ for all $i \in \{1, \ldots, N\}$. Using vector notation together with (1), this requirement is equivalent to:

$$\vec{\lambda}(I - P)^{-1} < \vec{\mu}$$

This simple matrix inequality is all that is required to check stability for Jackson networks. It shows which rate vectors $\vec{\lambda}$ can be supported in a given network, and likewise shows how to design the server rates $\vec{\mu}$ to handle a given set of traffic.

A. The Steady State Distribution for Jackson Networks

Now assume that $\gamma_i < \mu_i$ for all $i \in \{1, \ldots, N\}$, and define $\rho_i = \gamma_i/\mu_i$. By Little's Theorem, ρ_i is the time average fraction of time that node i is busy. Remarkably, the steady state joint probability distribution can be obtained as the product of marginal distributions, where the marginal distribution for each node i has the form of an $M/M/1$ queue with loading ρ_i. That is, we have the following theorem:

Theorem 1: (Jackson Theorem) In a Jackson network with $\rho_i < 1$ for all $i \in \{1, \ldots, N\}$ (where $\rho_i = \gamma_i/\mu_i$, with γ_i defined in (1)), the joint steady state distribution for the number of packets in each node is given by:

$$Pr[L_1 = n_1, L_2 = n_2, \ldots, L_3 = n_3] = \Pi_{i=1}^{N}(1 - \rho_i)\rho_i^{n_i}$$

It is important to note that the proof does *not* follow from Burke's theorem. Burke's theorem can be used to show the above statement only in the special case of feedforward networks (with no feedback). However, if one then guesses that the steady state probability distribution has the same form (given above) for networks with feedback, this guess can be verified by showing the resulting probabilities satisfy the Global Balance Equations for a continuous time Markov chain (one *cannot use the detail equations* to show this, as the multi-dimensional Markov chain is not reversible). We further emphasize that in systems with feedback, each individual queue is not necessarily an $M/M/1$ queue (the aggregate arrival process to the queue may not even be Poisson if there is feedback in the network). However, the steady state distribution for each queue just happens to have the same distribution as that of a corresponding $M/M/1$ queue with the same loading. As a corollary to Theorem 1, it also follows that the average number of packets and average delay in each queue is the same as the corresponding $M/M/1$ queue:

$$\begin{aligned} \overline{L_i} &= \frac{\rho_i}{1 - \rho_i} \text{ for all } i \in \{1, \ldots, N\} \\ \overline{W}_i &= \frac{1}{\mu_i - \gamma_i} \text{ for all } i \in \{1, \ldots, N\} \end{aligned}$$

Therefore, the total average number of packets in the system and the total average end-to-end delay is given by:

$$\begin{aligned} \overline{L}_{tot} &= \sum_{i=1}^{N} \overline{L}_i \\ \overline{W} &= \frac{\overline{L}_{tot}}{\sum_{i=1}^{N} \lambda_i} \end{aligned}$$

APPENDIX A — WHEN IS THE $(I - P)$ MATRIX INVERTIBLE?

The manipulations in Section I implicitly assume that the $(I - P)$ matrix is invertible. It turns out that checking if this matrix is invertible requires only a simple thought experiment. Imagine an empty network, and inject a single packet at a particular source node. The packet bounces around the network according to the transition probability matrix $P = (P_{ij})$. Note that when the packet is in a certain node i, the probability it leaves the network is given by $1 - \sum_j P_{ij}$.[1] The $(I - P)$ matrix is invertible whenever any such single packet eventually leaves the network (with probability 1), regardless of which source node it entered the network from. This is described in the following theorem.

Theorem 2: The $(I - P)$ matrix is invertible whenever $P^n \rightarrow (0)$, which occurs whenever any single packet that enters the network will eventually leave. Furthermore, in such cases the inverse is equal to:

$$(I - P)^{-1} = I + P + P^2 + P^3 + \ldots = I + \sum_{n=1}^{\infty} P^n$$

Proof: By a simple telescoping series argument we have the following identity:

$$(I - P)(I + P + P^2 + \ldots + P^n) = I - P^{n+1} \tag{2}$$

[1]Note that, in general, the row sums of P are less than or equal to 1, but are strictly less than 1 for a particular row i in the case when a packet can exit the network from node i.

It is not difficult to show that if $P^n \to 0$, then it must decrease geometrically, in which case the infinite sum $\sum_{n=1}^{\infty} P^n$ converges to a finite matrix. Thus, assuming that $P^n \to 0$, we take limits of (2) to obtain:

$$(I - P)\left(I + \sum_{n=1}^{\infty} P^n\right) = I$$

which implies that $(I - P)$ is invertible, and that $(I - P)^{-1} = (I + \sum_{n=1}^{\infty} P^n)$.

Now let $\vec{e}_i$ represent a unit row vector with a 1 in the i^{th} entry, and a 0 in every other entry. Considering the single packet thought experiment, it follows that if the packet is injected into the network at source node i, the probability the packet is in node j after n transitions is given by the j^{th} entry of the row vector $\vec{e}_i P^n$. But this converges to zero for every source node i if and only if P^n converges to the all-zero matrix as $n \to \infty$. $\square$