# Final Exam — EE 549
# Friday May 9, 2008
# 8:00-10:00am, MHP B7B

## INSTRUCTIONS

- This exam lasts for 2 hours.
- This exam is closed book and closed notes. No Calculators or laptops allowed.
- Put your name at the top corner of this page. If you are a DEN student, put "DEN" in parentheses.

## QUESTIONS

There are four questions, with point values given below:

1) Little's Theorem and Priority Service (25 points)
2) Express Lanes for Grocery Markets, And Vacations (25 points)
3) Network Calculus (25 points)
4) Emptying Times, Jackson Networks, and Deterministic Service Time Trees (25 points)

## FORMULAS

1) I-O Equation:
   - For any $t_1 \leq t_2$: $U(t_2) = U(t_1^-) + X[t_1, t_2] - Y[t_1, t_2]$
2) Leaky Bucket Formulas:
   $X(t) \sim (r, \sigma)$, $\mu(t) \sim (\overline{\mu}, \gamma)$. If $r \leq \overline{\mu}$ then:
   - $U(t) \leq \sigma + \gamma$
   - $Y(t) \sim (r, \sigma + \gamma)$
   - $Delay_{FIFO} \leq \frac{\sigma+\gamma}{\overline{\mu}}$
3) Two leaky bucket inputs $X_1(t), X_2(t)$ (splitting property):
   $X_1(t) \sim (r_1, \sigma_1), X_2(t) \sim (r_2, \sigma_2)$, $\mu(t) \sim (\overline{\mu}, \gamma)$. If $r_1 + r_2 \leq \overline{\mu}$, then:
   - $Y_1(t) \sim (r_1, \sigma_1 + \sigma_2 + \gamma)$
   - $Y_2(t) \sim (r_2, \sigma_1 + \sigma_2 + \gamma)$
   - $Y(t) = Y_1(t) + Y_2(t) \sim (r_1 + r_2, \sigma_1 + \sigma_2 + \gamma)$
4) GI/B/1 Queue: $\overline{L} = \frac{\lambda + \mathbb{E}\{A^2\} - 2\lambda^2}{2(\mu - \lambda)}$
5) B/B/1 Queue: $\overline{L} = \frac{\lambda(1-\lambda)}{\mu - \lambda}$
6) M/M/1 Queue: ($\rho = \lambda/\mu$)
   - $\overline{L} = \rho/(1-\rho)$
   - $Pr[L = i] = (1-\rho)\rho^i$ for $i \in \{0, 1, 2, \ldots\}$
7) Little's Theorem: $\lambda\overline{W} = \overline{L}$
8) $M/G/1$ Queue: $\overline{W}_q = \frac{\overline{R}}{1-\rho} = \frac{\lambda\mathbb{E}\{S^2\}}{2(1-\rho)}$
9) $M/D/1$ Queue: $\overline{L} = \rho + \frac{\rho^2}{2(1-\rho)}$
10) Residual Life:
    - Triangles arriving with rate $\lambda$: $\overline{R} = \frac{\lambda\mathbb{E}\{S^2\}}{2}$.
    - Triangles arriving back-to-back: $\overline{R} = \frac{\mathbb{E}\{S^2\}}{2\mathbb{E}\{S\}}$.

## I. Little's Theorem and Priority Service (25 points)

a) State Little's Theorem. Be sure to define your notation.

b) Consider a single-server queue with two packet arrival streams: The first stream has packet arrivals of rate $\lambda_1$ packets/sec, and has i.i.d. service times with first and second moments $\mathbb{E}\{S_1\}$ and $\mathbb{E}\{S_1^2\}$. The second has packet arrivals of rate $\lambda_2$ packets/sec, and has i.i.d. service times with first and second moments $\mathbb{E}\{S_2\}$ and $\mathbb{E}\{S_2^2\}$ (see Fig. 1). Assume service is FIFO, and the time average fraction of time the system is busy is strictly less than 1.

State the exact time average fraction of time the *server is busy serving type 1 data*, and prove your result. You can use Little's Theorem if you like, and you can assume appropriate time averages are well defined. But you must present a complete proof. *If you are not sure you have a complete proof, imagine me writing the question "why?" as I grade your answer (and deducting points if the "why" is not clearly answered). Note that "type 1 data" is the same as "stream 1 data."*

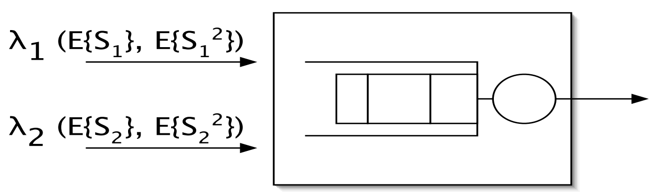


Fig. 1. Two input streams to a single-server FIFO queue.

c) Consider the single-server queueing system of Fig. 2. There are two inputs, the first has *Poisson* packet arrivals of rate $\lambda_1$ packets/sec, with i.i.d. service times with moments $\mathbb{E}\{S_1\}$ and $\mathbb{E}\{S_1^2\}$. The second is *non-Poisson* with i.i.d. service times with moments $\mathbb{E}\{S_2\}$ and $\mathbb{E}\{S_2^2\}$.

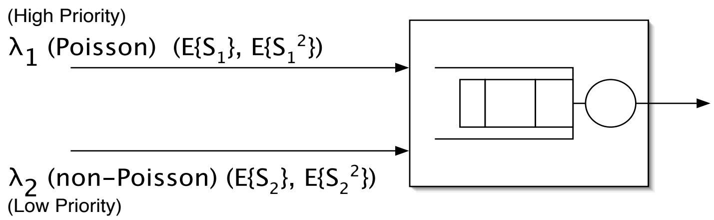


Fig. 2. Two input streams to a single-server queue with non-preemptive priority service.

Assume that type 1 data (i.e., stream 1 data) has *non-preemptive priority over type 2 data*, so that it cannot preempt a type 2 packet in the server, but type 1 packets are always placed into the server when the server finishes (if there are type 1 packets available). Packets are served in FIFO order within their own priority class. Suppose an observer measures the time average number of packets in the system (including both type 1 and type 2 packets, and including both the queue buffer and the server) and finds that it is $\overline{L} = 42.1$ packets.

Compute the exact average delay of the *low priority (type 2) packets* in the system (including both queueing delay and service time).

## II. EXPRESS LANES FOR GROCERY MARKETS, AND VACATIONS

Customers arrive to a check-out counter at a grocery market according to a Poisson process of rate $\lambda$ customers/sec. Customers have i.i.d. service times $\{S_i\}$ that are *uniformly distributed between* $0$ *and* $T$ *seconds* (so that the service density is $p_S(t) = 1/T$ for $0 \leq t \leq T$). There are two lanes: One is an express lane where customers with small service times are placed. Specifically, a customer $i$ with service time $S_i \leq x$ is placed in lane 1, where $x$ is a constant that satisfies $0 < x < T$. The other is a regular lane where all other customers are placed (see Fig. 3). Each lane represents a single-server queue.

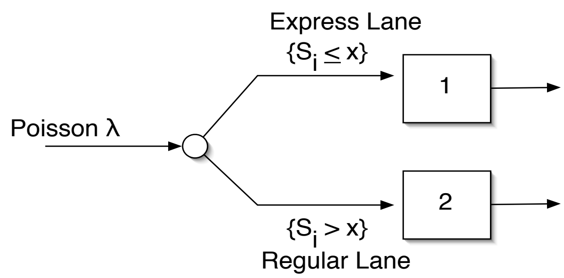


Fig. 3. An illustration of the express lane system.

a) State a necessary and sufficient condition for stability of the express lane (lane 1).

b) Assuming stability, compute $\overline{W}_1$, the average delay for express lane customers (including both queueing and service time).

c) Consider now a new grocery market with only one lane, represented by a single-server queue. Customers arrive according to a Poisson process of rate $\lambda$, and have i.i.d. service times with moments $\mathbb{E}\{S\}$ and $\mathbb{E}\{S^2\}$.

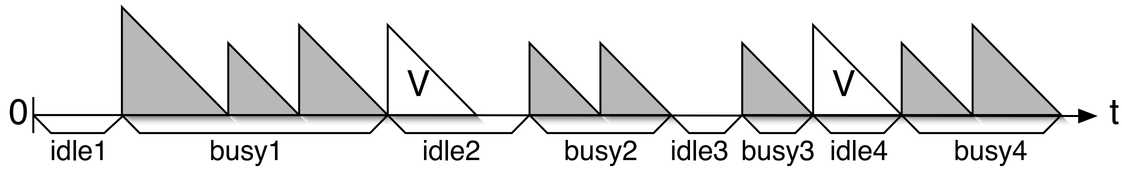


Fig. 4. A timeline of events for the checkout counter where the worker is allowed one vacation at the end of every odd busy period.

The check-out worker is allowed to take a single vacation of constant size $V$ seconds *at the end of every odd busy period*. At the end of the vacation, the worker must go back to the check-out counter and either waits for the next customer (if no customers arrived during the vacation) or immediately starts serving the next customer (if there were one or more arrivals during the vacation). A typical timeline of events is shown in Fig. 4. Compute $\overline{W}_q$, the average waiting time in the queue buffer (not including service time). *Hint: Consider a renewal event to occur at time* $0$ *and again at the end of every even busy period.*

## III. Network Calculus (25 Points)

a) Give the definition of a leaky bucket arrival process $X(t)$ with rate $r$ bits/sec and burst $\sigma$ bits, i.e., $X(t) \sim (r, \sigma)$.

b) Let $X(t) \sim (r, \sigma)$, and suppose all packets have fixed size $B$ kilobits (where $B \leq \sigma$). Suppose we split the process into two streams by *independent probabilistic routing*, so that every packet is independently placed to stream 1 with probability $1/4$, and else it is placed in stream 2 (see Fig. 5).

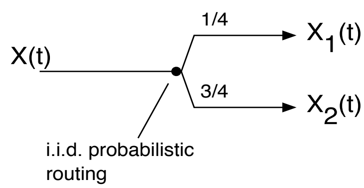


Fig. 5. An illustration of the i.i.d. probabilistic splitting for Problem IIIb.

Let $X_1(t)$ and $X_2(t)$ represent the resulting processes consisting of packets placed to streams 1 and 2, respectively, so that $X_1(t) + X_2(t) = X(t)$. Give the tightest possible leaky bucket parameters $(r_1, \sigma_1)$ you can for the $X_1(t)$ process. Give two or three sentences (possibly containing equations) that explain your answer.

c) Consider a single-server work conserving queue with input $X(t) \sim (r, \sigma)$ and transmission rate $\mu(t) \sim (\overline{\mu}, \gamma)$. Prove that if the queue is initially empty and if $r \leq \overline{\mu}$, then the unfinished work $U(t)$ satisfies $U(t) \leq \sigma + \gamma$ for all $t$.

d) Consider the multi-hop network with leaky bucket inputs as shown in Fig. 6.

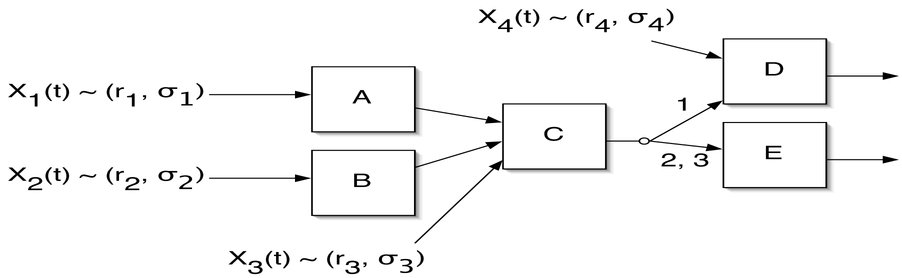


Fig. 6. The Multi-Hop Network (with FIFO) for Problem IIId.

Assume service is FIFO and that we have a fluid departure model. Assume arrival processes are given by $X_i(t) \sim (r_i, \sigma_i)$ for $i \in \{1, 2, 3, 4\}$. The data streams $X_1(t), X_2(t), X_3(t)$ each take different routes after node C:

- Stream 1 data enters node $D$.
- Stream 2 and Stream 3 data enters node $E$.

The fourth process $X_4(t)$ enters node $D$ (along with the stream 1 data that exits node $C$). Assume processing rates are constant at each node and given by $\mu_A, \mu_B, \mu_C, \mu_D, \mu_E$. Further assume that:

$$r_1 \leq \mu_A \;\; , \;\; r_2 \leq \mu_B \;\; , \;\; r_1 + r_2 + r_3 \leq \mu_C \;\; , \;\; r_1 + r_4 \leq \mu_D \;\; , \;\; r_2 + r_3 \leq \mu_E$$

Compute the tightest possible worst-case queueing bounds for $U_A(t)$, $U_B(t)$, $U_C(t)$, $U_D(t)$, and $U_E(t)$ using the network calculus results from this course. *Justify your answers where appropriate.*

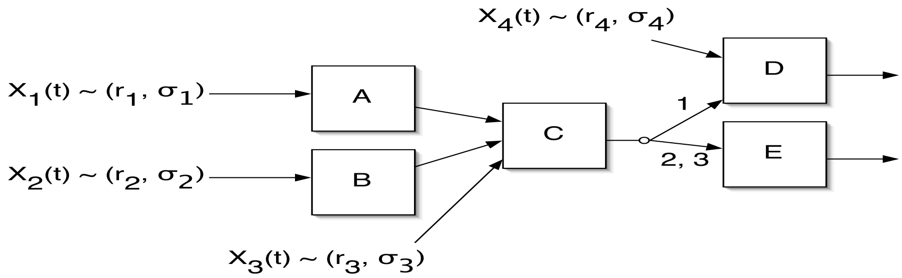


Fig. 7. The same Multi-Hop Network (with FIFO) from part (d) on previous page.

e) For the same multi-hop network with FIFO as in part (d) (illustrated again above), compute the tightest possible worst case end-to-end delay for data of stream $X_1(t)$ (including the delay in all nodes that it visits).

## IV. Emptying Times, Jackson Networks, and Deterministic Service Time Trees (25 points)

a) Consider the following network with three queues. Service times in all queues are i.i.d. exponential with rate $\mu$. There are two packets in queue $A$ at time zero, and no packets in queues $B$ and $C$ (so that the system state is $(L_A, L_B, L_C) = (2, 0, 0)$). No new packets ever arrive. After service in node $A$, each packet is independently placed in either queue $B$ or $C$, with equal probability. Compute the expected time until the system empties.

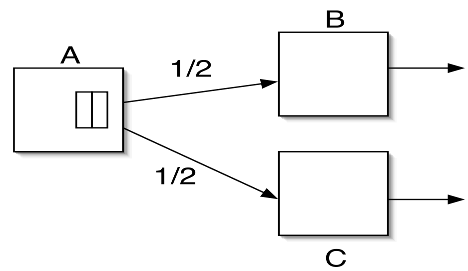


Fig. 8. A three queue system (with queues A, B, C) with two packets initially in node A. All service times are exponential with rate $\mu$.

b) Consider the 3-queue Jackson network in Fig. 9. Service in all queues is i.i.d. exponential with rate $\mu$. Packets departing node $A$ independently go to either node $B$ or $C$, with equal probability. Packets departing node $C$ independently either exit the system or go to node $B$ (with equal probability). Packets departing node $B$ independently either exit the system or go to node $A$ (with equal probability).

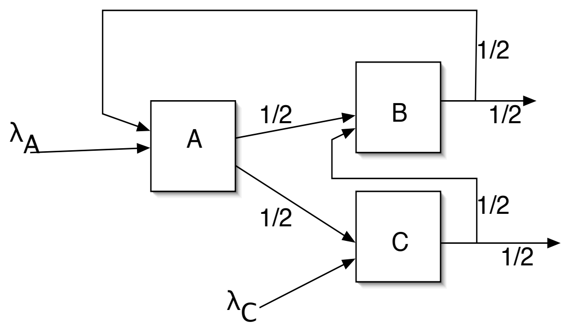


Fig. 9. A three queue Jackson Network. All service is i.i.d. exponential with rate $\mu$.

Assuming stability, write 3 equations and 3 unknowns that can be used to solve for the aggregate input rate into node $A$ (*do not bother to solve these equations*).

c) In terms of your notation in part (b), write the steady state probability that there are 5 packets in node $A$ for the Jackson network of Fig. 9 (*again note that you do not need to solve your equations in part (b)*).

We have the following theorem for deterministic service time queues with constant and equal service times $T$:

*Theorem 1:* Compare the systems $I$ and $II$ in Fig. 10. System $I$ has two queues, and System $II$ is the same as $I$ with the exception that the first queue (queue $A$) is replaced by a pure delay line of duration $T$. If both systems are initially empty, then for all $t$ we have:

$$D(t) = \tilde{D}(t)$$

that is, the departure processes are exactly the same for all time.

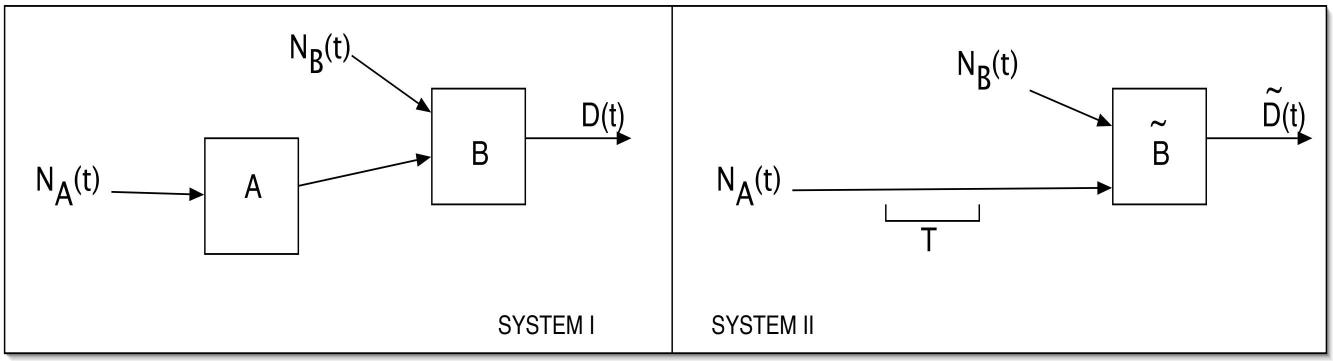


Fig. 10. The equivalent model theorem.

d) Consider the 3-node system of Fig. 11. All arrival processes are independent and Poisson with rates $\lambda_A, \lambda_B, \lambda_C$. All service time is constant and equal to $T$ seconds. Assume that $(\lambda_A + \lambda_B + \lambda_C)T < 1$. Compute the exact average number of packets in node $C$ (including both packets in the queueing buffer and in the server).

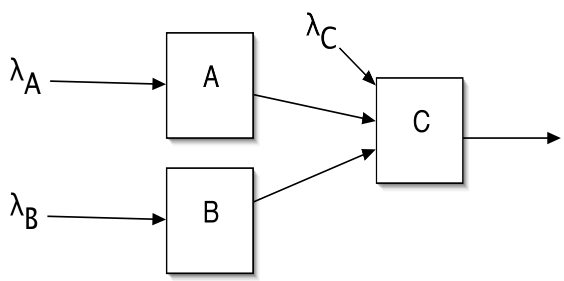


Fig. 11. A 3-node deterministic service time tree network with independent Poisson inputs.