Midterm Exam 2 — EE 549
Wednesday, April 16, 2008

ON-CAMPUS TIME AND LOCATION

- Wednesday, April 16, 9:30-10:55am, OHE 100C

INSTRUCTIONS

- This exam lasts for 85 minutes.
- This exam is closed book and closed notes. No book, notes, calculators, laptops, or cell phones allowed.
- Put your name at the top corner of this page. If you are a DEN student, put "DEN" in parentheses.

QUESTIONS

There are three questions, with point values given below:

1) IMET with Sub-Optimal Computation (38 points)

2) Discrete Time Markov Chains (34 points)

3) Queueing with Discrete Time Markov Chains (32 points)

FORMULAS

1) I-O Equation:
   - For any $t_1 \leq t_2$: $U(t_2) = U(t_1^-) + X[t_1, t_2] - Y[t_1, t_2]$

2) Leaky Bucket Formulas:
   $X(t) \sim (r, \sigma)$, $\mu(t) \sim (\overline{\mu}, \gamma)$. If $r \leq \overline{\mu}$ then:
   - $U(t) \leq \sigma + \gamma$
   - $Y(t) \sim (r, \sigma + \gamma)$
   - $Delay_{FIFO} \leq \frac{\sigma+\gamma}{\overline{\mu}}$

3) Two leaky bucket inputs $X_1(t), X_2(t)$ (splitting property):
   $X_1(t) \sim (r_1, \sigma_1), X_2(t) \sim (r_2, \sigma_2)$, $\mu(t) \sim (\overline{\mu}, \gamma)$. If $r_1 + r_2 \leq \overline{\mu}$, then:
   - $Y_1(t) \sim (r_1, \sigma_1 + \sigma_2 + \gamma)$
   - $Y_2(t) \sim (r_2, \sigma_1 + \sigma_2 + \gamma)$
   - $Y(t) = Y_1(t) + Y_2(t) \sim (r_1 + r_2, \sigma_1 + \sigma_2 + \gamma)$

4) GI/B/1 Queue: $\overline{L} = \frac{\lambda + \mathbb{E}\{A^2\} - 2\lambda^2}{2(\mu - \lambda)}$

5) B/B/1 Queue: $\overline{L} = \frac{\lambda(1-\lambda)}{\mu - \lambda}$

6) M/M/1 Queue: $(\rho = \lambda/\mu)$
   - $\overline{L} = \rho/(1-\rho)$
   - $Pr[L = i] = (1-\rho)\rho^i$ for $i \in \{0, 1, 2, \ldots\}$

7) Little's Theorem: $\lambda\overline{W} = \overline{L}$

I. IMET WITH SUB-OPTIMAL COMPUTATION (38 POINTS)

Consider the wireless downlink problem with K channels. Each channel i has a concave rate-power curve. Recall that $T_{min}(\vec{U})$ represents the minimum time to clear a backlog vector $\vec{U}$, optimized over all possible power allocation policies that satisfy the power constraint. In class, we proved the following properties:

Lemma 1: (Properties of $T_{min}(\vec{U})$):

(1) $T_{min}(\vec{U}_1) \leq T_{min}(\vec{U}_2)$ whenever $\vec{U}_1 \leq \vec{U}_2$.

(2) $T_{min}(\vec{U}_1 + \vec{U}_2) \leq T_{min}(\vec{U}_1) + T_{min}(\vec{U}_2)$.

(3) $T_{min}(\vec{r}t) \leq t$ whenever $\vec{r} \in \Lambda$, where Λ is the downlink capacity region, and t is any time that satisfies $t \geq 0$.

a) Write a paragraph to explain why property 1 in Lemma 1 is true. You should write at least two or three sentences of explanation.

b) Write a paragraph to explain whey property 2 in Lemma 1 is true. You should write at least two or three sentences of explanation.

Lemma 1 (repeated from previous page): (Properties of $T_{min}(\vec{U})$):
(1) $T_{min}(\vec{U}_1) \leq T_{min}(\vec{U}_2)$ whenever $\vec{U}_1 \leq \vec{U}_2$.
(2) $T_{min}(\vec{U}_1 + \vec{U}_2) \leq T_{min}(\vec{U}_1) + T_{min}(\vec{U}_2)$.
(3) $T_{min}(\vec{r}t) \leq t$ whenever $\vec{r} \in \Lambda$, where Λ is the downlink capacity region, and t is any time that satisfies $t \geq 0$.

c) Suppose the wireless downlink system has inputs $X_i(t)$ that are leaky bucket with parameters (r_i, σ_i). Let $\vec{r} = (r_1, \ldots, r_K)$, and $\vec{\sigma} = (\sigma_1, \ldots, \sigma_K)$. Suppose we implement IMET with fixed frames of size of T. However, rather than using the *optimal* minimum emptying time algorithm for $T_{min}(\vec{U})$, we use an algorithm that allocates power to clear in time $T^*(\vec{U})$, where:

$$T^*(\vec{U}) \leq \left(\frac{1}{\gamma}\right) T_{min}(\vec{U}) + z$$

for some factor γ that satisfies $0 < \gamma \leq 1$, and some constant time z such that $z \geq 0$. Let ρ be a value such that $0 < \rho < 1$. Suppose that:

$$\vec{r} \in \gamma\rho\Lambda \quad \text{(and hence } (\tfrac{r_1}{\gamma\rho}, \tfrac{r_2}{\gamma\rho}, \ldots, \tfrac{r_K}{\gamma\rho}) \in \Lambda)$$

Design a frame size T such that this modified IMET (with sub-optimal $T^*(\vec{U})$ scheduling) never drops any data and ensures worst case delay is at most $2T$. *Hint: You should use the three properties of $T_{min}(\vec{U})$ given in Lemma 1, repeated above for convenience. Note that these properties hold for $T_{min}(\vec{U})$ but may not hold for $T^*(\vec{U})$.*

Fact (from part (c)): If $T^*(\vec{U}) \leq \left(\frac{1}{\gamma}\right) T_{min}(\vec{U}) + z$, then IMET never drops data provided that:

- $\vec{r} \in \gamma\rho\Lambda$
- The frame size T satisfies $T \geq f(z, \gamma)$, where $f(z, \gamma)$ is a function determined in part (c).

Part (c) asked you to compute the function $f(z, \gamma)$ in the above fact. In this question, you should use the fact above with the general notation $f(z, \gamma)$. Thus, please write your answer in terms of the $f(z, \gamma)$ function (you do not need to know what this function is, just in case your answer to part (c) is incorrect).

d) Consider now a system that uses IMET, but the beginning of each frame requires δ units of time to compute the solution to $T_{min}(\vec{U})$. Thus, during any frame, we first are *idle* for δ units of time (while we are computing $T_{min}(\vec{U})$, where $\vec{U}$ is the backlog at the beginning of the frame), and then we implement this solution for the rest of the frame. Define a function $T^*(\vec{U})$ to be the time required to clear a backlog vector $\vec{U}$ under this policy. Suppose that $\vec{r} \in \rho\Lambda$ (where $0 < \rho < 1$). Use the above fact to give a required frame size T for which this policy never drops data.

II. DISCRETE TIME MARKOV CHAINS (34 POINTS)

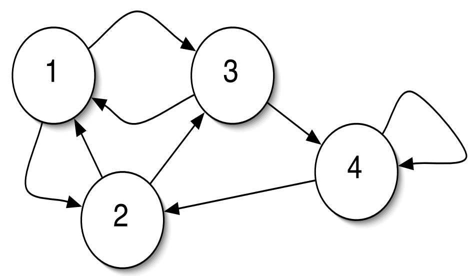

Fig. 1. A 4-state DTMC. The arrows represent positive transition probabilities $P_{ij} > 0$.

a) Consider the 4-state DTMC shown in Fig. 1. Suppose all arrows in the figure above represent positive transition probabilities $P_{ij} > 0$. Suppose the DTMC is irreducible, aperiodic, and has a steady state probability distribution given by $\vec{\pi} = (\pi_1, \pi_2, \pi_3, \pi_4)$. Prove that $\pi_1 > 0$. (*Hint: assume* $\pi_1 = 0$*, and reach a contradiction*).

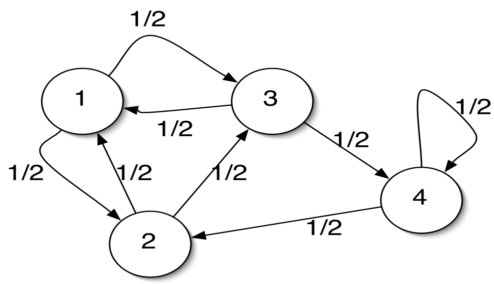

Fig. 2. A 4-state DTMC.

b) Consider the 4-state DTMC of Fig. 2. The chain is irreducible, aperiodic, and has a steady state distribution $\vec{\pi}$ that satisfies the stationary equation. Find the stationary distribution $\vec{\pi} = (\pi_1, \pi_2, \pi_3, \pi_4)$. You should justify your derivations with (brief) explanations.

III. DISCRETE TIME QUEUES (32 POINTS)

Consider a discrete time queue with input process $A(t)$ and server process $\mu(t)$, for $t \in \{0, 1, 2, \ldots\}$. Assume $A(t)$ and $\mu(t)$ are in units of packets, and $A(t) \in \{0, 1\}$, $\mu(t) \in \{0, 1\}$, so that at most one packet can arrive on any slot, and at most one packet can be served on any slot. The dynamic queueing equation is given by:

$$L(t+1) = \max[L(t) - \mu(t), 0] + A(t)$$

Assume that $A(t)$ is i.i.d. Bernoulli with rate λ. Assume that $\mu(t)$ is Markov modulated as follows: Let $Z(t)$ be the 3-state Markov chain with states a, b, c and transition probabilities ϵ and δ, as shown in Fig. 3. The process $\mu(t)$ is given by:

$$\mu(t) = \begin{cases} 1 & \text{if } Z(t) = c \\ 0 & \text{otherwise (i.e., if } Z(t) = a \text{ or } Z(t) = b) \end{cases}$$

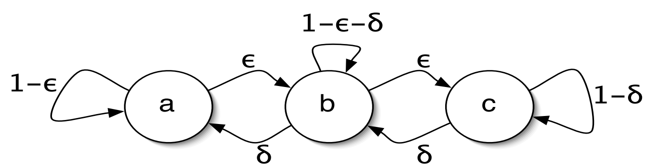

Fig. 3. The 3-state DTMC $Z(t)$ that affects the server process $\mu(t)$.

Assume that all transition probabilities labeled with arrows in the above DTMC for $Z(t)$ are positive (i.e., $\epsilon > 0$, $\delta > 0$, $1 - \epsilon > 0$, $1 - \delta > 0$, $1 - \epsilon - \delta > 0$).

a) Draw the Markov chain that represents this queueing system. You should clearly draw the states, and should clearly draw arrows between states if and only if there is a non-zero transition probability. *To avoid mess, do not label the arrows with their transition probability...I just want to see the states and the arrows, and will grade based on whether or not there are any states or arrows that are missing or incorrect. There is another page if you mess up and want to start again. There is also a part (b) question on the next page.*

b) Go back to your figure on the previous page. Now pick any single state you want. Label all outgoing transition probabilities P_{ij} of that state with their exact value. To avoid mess and to make it easy for me to grade, do not label any transition probabilities other than those going out of the single state that you choose.

Thus, you can now write on the picture you drew on the previous page. If you mess up your picture on the previous page and want to draw another picture for the answer for both (a) and (b) on this page, feel free to do that. But let me know which picture I should grade.