EE549: Problem Set #8 Solutions

I. A TANDEM OF QUEUES WITH BERNOULLI SERVICE

a) The first queue is $B/B/1$ with average number of packets:

$$\overline{L}_1 = \frac{\lambda(1-\lambda)}{\mu_1 - \lambda}$$

In steady state, the departure process of the first queue is i.i.d. Bernoulli with rate λ. The second queue is thus $B/B/1$ with average number of packets:

$$\overline{L}_2 = \frac{\lambda(1-\lambda)}{\mu_2 - \lambda}$$

b) In steady state, the departure process of the second queue is i.i.d. Bernoulli with rate λ. Let $A_1(t)$ represent this departure process, and let $A_2(t)$ represent the additional (independent) input to queue 3. The total arrival process to queue 3 is given by:

$$A(t) = A_1(t) + A_2(t)$$

where $A_1(t)$ and $A_2(t)$ are independent and i.i.d. Bernoulli processes with rates λ and λ_2, respectively. This third queue is thus a discrete time $GI/B/1$ queue, with arrival moments:

$$\begin{aligned}
\mathbb{E}\{A(t)\} &= \lambda + \lambda_2 \\
\mathbb{E}\{A(t)^2\} &= \mathbb{E}\{A_1(t)^2 + 2A_1(t)A_2(t) + A_2(t)^2\} \\
&= \mathbb{E}\{A_1(t)\} + \mathbb{E}\{A_2(t)\} + 2\mathbb{E}\{A_1(t)\}\mathbb{E}\{A_2(t)\} \\
&= \lambda + \lambda_2 + 2\lambda\lambda_2
\end{aligned}$$

If $\lambda + \lambda_2 \geq \mu_3$, then the average number of packets in this third queue is ∞. Let us assume that $\lambda + \lambda_2 < \mu_3$. The average number of packets in this third queue is thus:

$$\overline{L}_3 = \frac{\mathbb{E}\{A\} + \mathbb{E}\{A^2\} - 2\mathbb{E}\{A\}^2}{2(\mu_3 - \mathbb{E}\{A\})} = \frac{\lambda + \lambda_2 + (\lambda + \lambda_2 + 2\lambda\lambda_2) - 2(\lambda+\lambda_2)^2}{2(\mu_3 - \lambda - \lambda_2)} = \frac{\lambda + \lambda_2 - (\lambda^2 + \lambda_2^2)}{\mu_3 - \lambda - \lambda_2}$$

II. RANDOMIZED WIRELESS SCHEDULING

a) Note that the second queue is not $B/B/1$ because we cannot have both a departure and an arrival on the same slot (arrival events and service opportunities are mutually exclusive). However, events are still i.i.d. over slots. The input process is i.i.d. Bernoulli, and so arrivals occur i.i.d. with probability λ, and the service opportunities are mutually exclusive of arrivals and i.i.d. over slots with probability p_2. The Markov chain is given in Fig. 1:

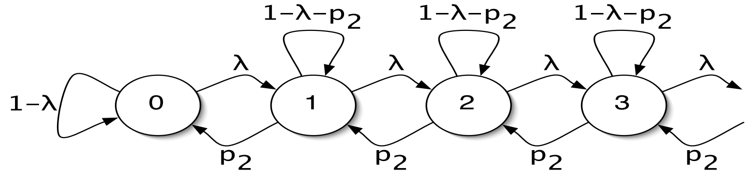

Fig. 1. The Markov chain for the second queue of Problem 2.a.

b) Let π_i be the steady state probability that the number of packets is i. Because a steady state exists, we have by the cut-set equations (define $\rho = \lambda/p_2$):

$$\begin{aligned} \pi_0\lambda = \pi_1 p_2 \quad &\Longrightarrow \quad \pi_1 = \pi_0\rho \\ \pi_1\lambda = \pi_2 p_2 \quad &\Longrightarrow \quad \pi_2 = \pi_0\rho^2 \\ \cdots \quad &\Longrightarrow \quad \cdots \\ \pi_i\lambda = \pi_{i+1} p_2 \quad &\Longrightarrow \quad \pi_i = \pi_0\rho^i \end{aligned}$$

Therefore, $\pi_0 = (1-\rho)$, and $\pi_i = (1-\rho)\rho^i$ for $i \in \{0, 1, 2, \ldots\}$. Hence:

$$\overline{L}_2 = \sum_{i=1}^{\infty}(1-\rho)\rho^i = \frac{\rho}{1-\rho} = \frac{\lambda}{p_2 - \lambda}$$

The average delay in the system is thus given by (using Little's Theorem):

$$\overline{W} = \frac{\overline{L}_1 + \overline{L}_2}{\lambda} = \frac{\lambda(1-\lambda)}{\lambda(p_1-\lambda)} + \frac{\lambda}{\lambda(p_2-\lambda)} = \frac{1-\lambda}{p_1-\lambda} + \frac{1}{p_2-\lambda}$$

Thus (using $p_2 = 1 - p_1$) we have:

$$\overline{W} = \frac{1-\lambda}{p_1-\lambda} + \frac{1}{1-p_1-\lambda}$$

c) Taking a derivative of $\overline{W}$ with respect to p_1 yields:

$$\frac{d\overline{W}}{dp_1} = \frac{-(1-\lambda)}{(p_1-\lambda)^2} + \frac{1}{(1-p_1-\lambda)^2}$$

Setting the derivative to zero yields:

$$(p_1-\lambda)^2 = (1-\lambda)(1-p_1-\lambda)^2$$

Thus:

$$p_1^2 - 2\lambda p_1 + \lambda^2 = (1-\lambda)((1-\lambda)^2 + p_1^2 - 2p_1(1-\lambda))$$

Let $p \triangleq p_1$. This leads to the quadratic equation:

$$\lambda p^2 + (-2\lambda + 2(1-\lambda)^2)p + (\lambda^2 - (1-\lambda)^3)$$

Thus:

$$p = \frac{(2\lambda - 2(1-\lambda)^2) + \sqrt{(2\lambda - 2(1-\lambda)^2)^2 - 4\lambda(\lambda^2 - (1-\lambda)^3)}}{2\lambda}$$

or

$$p = \frac{(2\lambda - 2(1-\lambda)^2) - \sqrt{(2\lambda - 2(1-\lambda)^2)^2 - 4\lambda(\lambda^2 - (1-\lambda)^3)}}{2\lambda}$$

Also note that we need $0 \leq p \leq 1$ and $\lambda < \min[p, 1-p]$. Thus, $p_1 = p$ and $p_2 = 1-p$. We need $\lambda < 1/2$ for finite average delay.

III. ON/OFF ARRIVALS

a) By Little's Theorem we have $Pr[L(t) = 0] = \lambda/\mu$, where $\lambda = \delta/(\epsilon + \delta)$. Thus:

$$\pi(0, ON) + \pi(0, OFF) = 1 - \rho \tag{1}$$

where $\rho = \lambda/\mu$. For the parameters given we have $\mu = 0.5$, $\lambda = 1/4 = 0.25$, and so $Pr[L(t) = 0] = 0.5$.

b) The 2-d Markov chain (same as previous problem set) is shown below:

Assume a steady state exists. We write convenient cut set equations: Drawing our cut to separate state $(0, OFF)$, we have:

$$\pi(0, OFF)\delta = \pi(1, OFF)(1-\delta)\mu \tag{2}$$

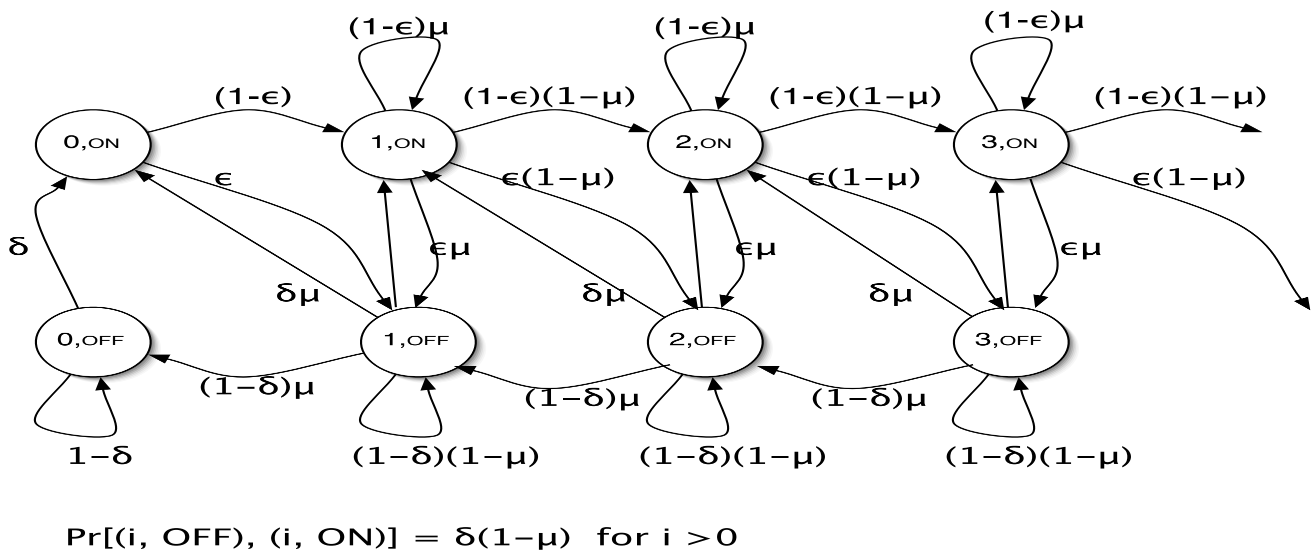

Fig. 2. The 2-dimensional Markov chain.

Drawing a cut separating states $(0, OFF)$ and $(0, ON)$ from the other states yields:

$$\pi(0, ON)(1) = \pi(1, OFF)\mu \tag{3}$$

Plugging (3) and (2) into (1) yields:

$$\pi(1, OFF)\mu + \pi(1, OFF)\frac{(1-\delta)\mu}{\delta} = 1 - \rho$$

and hence:

$$\pi(1, OFF) = (1-\rho)\delta/\mu$$

Therefore:

$$\pi(0, OFF) = (1-\rho)(1-\delta)$$

and:

$$\pi(0, ON) = (1-\rho)\delta$$

For $\mu = 0.5$, $\lambda = 0.25$, $\delta = 0.1$, $\epsilon = 0.3$, $\rho = 0.5$, we have:

$$\pi(0, ON) = (1-\rho)\delta = 0.5(0.1) = 0.05$$

c) Taking a cut between states for i and $i+1$ we have (for $i \geq 1$):

$$\pi(i, ON)(1-\mu) = \pi(i+1, OFF)\mu \ \ (i \in \{1, 2, 3, \ldots\} \tag{4}$$

Furthermore, taking a cut around only state $i+1$ yields (for $i \in \{1, 2, 3, \ldots\}$):

$$\pi(i+1, OFF)(\mu + \delta(1-\mu)) = \pi(i, ON)\epsilon(1-\mu) + \pi(i+1, ON)\epsilon\mu + \pi(i+2, OFF)(1-\delta)\mu \tag{5}$$

Plugging (4) into (5) yields:

$$\pi(i, ON)\frac{1-\mu}{\mu}(\mu + \delta(1-\mu)) = \pi(i, ON)\epsilon(1-\mu) + \pi(i+1, ON)\epsilon\mu + (1-\delta)(1-\mu)\pi(i+1, ON)$$

Thus:

$$\frac{\pi(i+1, ON)}{\pi(i, ON)} = \frac{(1-\mu) + \delta(1-\mu)^2/\mu - \epsilon(1-\mu)}{\epsilon\mu + (1-\delta)(1-\mu)} \triangleq \gamma$$

Therefore:

$$\pi(i, ON) = \pi(1, ON)\gamma^{i-1} \quad \text{for } i \in \{1, 2, 3, \ldots\}$$

$$\pi(i+1, OFF) = \pi(i, ON)(1-\mu)/\mu = \pi(1, ON)\gamma^{i-1}(1-\mu)/\mu \quad \text{for } i \in \{1, 2, \ldots\}$$

We also know $\pi(0, OFF), \pi(0, ON), \pi(1, OFF)$ from part (b). It suffices to solve for $\pi(1, ON)$. Drawing a cut around this state alone, we have:

$$\begin{aligned}
\pi(1, ON)((1-\mu) + \epsilon\mu) &= \pi(2, OFF)\delta\mu + \pi(1, OFF)\delta(1-\mu) + \pi(0, ON)(1-\epsilon) \\
&= \pi(1, ON)(1-\mu)\delta + (1-\rho)\delta^2(1-\mu)/\mu + (1-\rho)\delta(1-\epsilon)
\end{aligned}$$

Therefore:

$$\pi(1, ON) = \frac{(1-\rho)\delta^2(1-\mu)/\mu + (1-\rho)\delta(1-\epsilon)}{(1-\mu)(1-\delta) + \epsilon\mu}$$

For $\mu = 0.5$, $\lambda = 0.25$, $\epsilon = 0.3$, $\delta = 0.1$, $\rho = 0.5$ we have:

$$\gamma = 2/3 \quad , \quad \pi(1, ON) = 1/15$$

Thus:

$$\begin{aligned}
Pr[L(t) = 7] &= \pi(7, ON) + \pi(7, OFF) \\
&= \pi(1, ON)\gamma^6 + \pi(1, ON)\gamma^5 \\
&= \pi(1, ON)\gamma^5(\gamma + 1) \\
&= \pi(1, ON)(2/3)^5(5/3) \\
&= (1/15)(2/3)^5(5/3) = (1/9)(2/3)^5 = 0.0146319
\end{aligned}$$