

EE549: Problem Set #9

Solutions

I. RACING EXPONENTIALS

a) $Pr[20 \text{ arrivals before first departure}] = \left(\frac{\lambda}{\lambda+\mu}\right)^{19}$

b)

$$\begin{aligned} Pr[\text{exactly 1 dep. before first 20 arrivals}] &= Pr[\text{first dep. after first arrival, then 19 other arrivals}] \\ &\quad + \sum_{k=2}^{19} Pr[\text{first dep. after } k\text{th arrival, then } 20-k \text{ other arrivals}] \\ &= \left(\frac{\mu}{\lambda+\mu}\right) \left(\frac{\lambda}{\lambda+\mu}\right)^{18} \\ &\quad + \sum_{k=2}^{19} \left(\frac{\lambda}{\lambda+\mu}\right)^{k-1} \left(\frac{\mu}{\lambda+\mu}\right) \left(\frac{\lambda}{\lambda+\mu}\right)^{20-k} \\ &= \left(\frac{\mu}{\lambda+\mu}\right) \left(\frac{\lambda}{\lambda+\mu}\right)^{18} + 18 \left(\frac{\mu}{\lambda+\mu}\right) \left(\frac{\lambda}{\lambda+\mu}\right)^{19} \end{aligned}$$

II. TIME TO EMPTY

a) $\frac{9}{2\mu} + \frac{1}{\mu}$.

b) The third transition takes place at an expected time $3/(2\mu)$. After this, there are either 0 packets in one queue and 3 in another, or 1 in one queue and 2 in another. Thus:

$$\mathbb{E}\{T_{empty}\} = \frac{3}{2\mu} + Pr(3,0) \frac{3}{\mu} + Pr(1,2) \left(\frac{1}{2\mu} + (1/2) \frac{2}{\mu} + (1/2) \left(\frac{1}{2\mu} + \frac{1}{\mu} \right) \right)$$

Also:

$$\begin{aligned} Pr(3,0) &= 2(1/2)^3 = 1/4 \\ Pr(1,2) &= 1 - Pr(3,0) = 3/4 \end{aligned}$$

c) Let $m_i = \mathbb{E}\{\text{Time to reach 0} \mid \text{start in state } i\}$. We want to compute m_1 .

$$\begin{aligned} m_1 &= \frac{1}{\lambda+\mu} + \frac{\lambda}{\lambda+\mu} m_2 \\ m_2 &= \frac{1}{\lambda+\mu} + \frac{\mu}{\lambda+\mu} m_1 + \frac{\lambda}{\lambda+\mu} m_3 \\ m_3 &= \frac{1}{\mu} + m_2 \end{aligned}$$

Therefore:

$$m_2 = \frac{1}{\lambda+\mu} + \frac{\mu}{\lambda+\mu} m_1 + \frac{\lambda}{\lambda+\mu} \left(\frac{1}{\mu} + m_2 \right)$$

Hence:

$$m_2 = \frac{\frac{1}{\lambda+\mu} + \frac{\mu}{\lambda+\mu} m_1 + \frac{\lambda/\mu}{\lambda+\mu}}{1 - \lambda/(\lambda+\mu)} = \frac{1}{\mu} + m_1 + \lambda/\mu^2 = m_1 + (\lambda+\mu)/\mu^2$$

Thus:

$$m_1 = \frac{1}{\lambda+\mu} + \frac{\lambda}{\lambda+\mu} (m_1 + (\lambda+\mu)/\mu^2)$$

Thus:

$$m_1 = \frac{\frac{1}{\lambda+\mu} + \lambda/\mu^2}{1 - \lambda/(\lambda+\mu)} = \frac{1}{\mu} + \lambda(\lambda+\mu)/\mu^3$$

III. TRANSIENT ANALYSIS OF A 2-STATE CTMC

a) $v_0 = \lambda, v_1 = \mu$.

$$\begin{aligned}\frac{d}{dt}p_0(t) &= -\lambda p_0(t) + p_1(t)\mu \\ \frac{d}{dt}p_1(t) &= -\mu p_1(t) + p_0(t)\lambda\end{aligned}$$

b) $\frac{d}{dt}p_0(t) = -\lambda p_0(t) + (1 - p_0(t))\mu = -(\lambda + \mu)p_0(t) + \mu$

c) The constant solution $p_0(t) = \mu/(\lambda + \mu)$ satisfies the ODE of (b).

d) Let $p_0(t) = Ae^{-\gamma t}$. Then $\frac{d}{dt}p_0(t) = -\gamma p_0(t)$. Thus, if $\gamma = (\lambda + \mu)$, we solve the homogeneous equation.

e) Let $p_0(t) = \mu/(\lambda + \mu) + Ae^{-(\lambda + \mu)t}$. This solves the ODE in part (b). We need to solve for A that satisfies the initial condition. $p_0(0) = \theta_0 = \mu/(\lambda + \mu) + A$. Thus $A = \theta_0 - \mu/(\lambda + \mu)$. It follows that:

$$\begin{aligned}p_0(t) &= \frac{\mu}{\lambda + \mu} + \left(\theta_0 - \frac{\mu}{\lambda + \mu}\right)e^{-(\lambda + \mu)t} \\ p_1(t) &= \frac{\lambda}{\lambda + \mu} - \left(\theta_0 - \frac{\mu}{\lambda + \mu}\right)e^{-(\lambda + \mu)t}\end{aligned}$$

f) $p_0(\infty) = \mu/(\lambda + \mu)$. $p_1(\infty) = \lambda/(\lambda + \mu)$.

IV. A TOKEN BUFFER SYSTEM WITH POISSON PACKET AND TOKEN ARRIVALS

a) If $L(t) > 0$, then there are packets waiting in the transport layer storage buffer. It follows that there cannot be any tokens in the token buffer (else, the first packet would not be waiting), and so $T(t) = 0$. Likewise, if $T(t) > 0$, then there are tokens and so there cannot be any packets in the storage buffer (else, the first packet would consume a token and then disappear).

b) See Fig. 1.

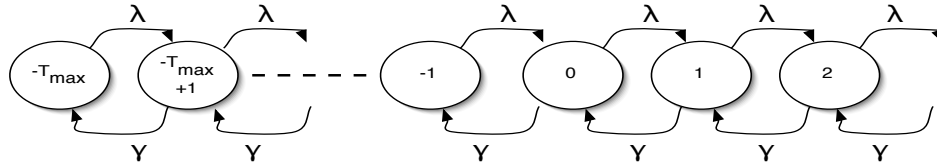


Fig. 1. The Markov chain $Z(t)$ for the token-bucket admission system.

c) Note that in steady state we have $Pr[L(t) = i] = (1 - \rho)\rho^{T_{max} + i}$ for $i \in \{0, 1, 2, \dots\}$, where $\rho = \lambda/\gamma$. Thus:

$$\mathbb{E}\{L\} = \sum_{i=0}^{\infty} i(1 - \rho)\rho^{T_{max} + i} = (1 - \rho)\rho^{T_{max}} \sum_{i=0}^{\infty} i\rho^i = \frac{\rho^{T_{max} + 1}}{1 - \rho}$$

Hence (by Little's Theorem):

$$\mathbb{E}\{W\} = \frac{1}{\lambda}\mathbb{E}\{L\} = \frac{\rho^{T_{max}}}{\gamma - \lambda}$$

V. RESIDUAL LIFE

a) The $e(t)$ function has triangles, like the $R(t)$ function, but the triangles are going up with a slope of 1. We have by renewal theory:

$$\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t e(\tau) d\tau = \frac{\mathbb{E}\{\text{triangle area}\}}{\mathbb{E}\{X\}} \text{ w.p.1} = \frac{\mathbb{E}\{X^2\}}{2\mathbb{E}\{X\}} \text{ w.p.1}$$

b) The time average residual time is the same as the time average elapsed time.

c) The size of the renewal period seen at time t is the sum of the elapsed time and the residual time seen at time t : $e(t) + R(t)$. The average seen can be obtained by a time average over all time:

$$\text{Expected size of renewal period seen} = \bar{e} + \bar{R} = \frac{\mathbb{E}\{X^2\}}{\mathbb{E}\{X\}}$$

d)

$$[\text{fraction of time } \tilde{X} \leq x] = \frac{\mathbb{E}\{X1_{X \leq x}\}}{\mathbb{E}\{X\}} = \frac{\mathbb{E}\{X | X \leq x\} Pr[X \leq x]}{\mathbb{E}\{X\}} = \frac{\int_0^x tp_X(t) dt}{\mathbb{E}\{X\}} \quad \text{w.p.1}$$

Differentiating yields (for $0 \leq x < \infty$):

$$\frac{d}{dx} [\text{fraction of time } \tilde{X} \leq x] = \frac{xp_X(x)}{\mathbb{E}\{X\}}$$

(Note that this indeed integrates to 1, as any density should). Therefore:

$$\mathbb{E}\{\tilde{X}\} = \int_0^\infty \frac{x(xp_X(x)) dx}{\mathbb{E}\{X\}} = \frac{\mathbb{E}\{X^2\}}{\mathbb{E}\{X\}}$$

This verifies part (c).

VI. M/G/1 QUEUES

a) Each queue is symmetric, and has Poisson arrivals with rate λ/K , and service times of constant time B/γ . This is just an M/G/1 queue.

$$\mathbb{E}\{W_q\} = \frac{\lambda(B/\gamma)^2}{2K(1-\rho)}$$

where $\rho = \lambda B/(K\gamma)$. Thus, the total average delay in the system (including service time) is:

$$\mathbb{E}\{W\} = \frac{\lambda(B/\gamma)^2}{2K(1-\rho)} + \frac{B}{\gamma}$$

b) This is a single M/G/1 queue with arrival rate λ , constant service times $B/(K\gamma)$. The value of ρ is thus the same. The total average delay is:

$$\mathbb{E}\{W\} = \frac{\lambda(B/(K\gamma))^2}{2(1-\rho)} + \frac{B}{K\gamma} = \frac{\lambda(B/\gamma)^2}{2K^2(1-\rho)} + \frac{B}{K\gamma}$$

This is exactly K times less than the delay in part (a).

c) Let $\lambda = \lambda_1 + \lambda_2 + \lambda_3$. The total arrival process is equivalent to a Poisson arrival process of rate λ , with i.i.d. packet service times X given by:

$$X = \begin{cases} B_1/\gamma & \text{with prob. } \lambda_1/\lambda \\ B_2/\gamma & \text{with prob. } \lambda_2/\lambda \\ B_3/\gamma & \text{with prob. } \lambda_3/\lambda \end{cases}$$

Therefore, this is just an M/G/1 queue, and:

$$\mathbb{E}\{W_q\} = \frac{\lambda \mathbb{E}\{X^2\}}{2(1-\rho)}$$

where:

$$\begin{aligned} \rho &= \lambda \mathbb{E}\{X\} = (\lambda_1 B_1 + \lambda_2 B_2 + \lambda_3 B_3)/\gamma \\ \mathbb{E}\{X^2\} &= (\lambda_1/\lambda)(B_1/\gamma)^2 + (\lambda_2/\lambda)(B_2/\gamma)^2 + (\lambda_3/\lambda)(B_3/\gamma)^2 \end{aligned}$$

VII. QUEUES WITH SET-UP TIME “VACATIONS”

a) Let \bar{I} , \bar{V} , and \bar{B} respectively represent the average duration of an idle, a set-up time, and a busy period. We know:

$$\bar{I} = 1/\lambda, \bar{V} = \mathbb{E}\{V\}$$

We also know that:

$$\frac{\bar{B}}{\bar{I} + \bar{V} + \bar{B}} = \rho$$

where $\rho = \lambda \mathbb{E}\{S\}$. Thus:

$$\bar{B} = \frac{\rho(\bar{I} + \bar{V})}{1 - \rho} = \frac{\rho(1/\lambda + \mathbb{E}\{V\})}{1 - \rho}$$

Thus:

- Fraction idle = $\frac{\bar{I}}{\bar{I} + \bar{V} + \bar{B}}$.
- Fraction set-up = $\frac{\bar{V}}{\bar{I} + \bar{V} + \bar{B}}$.
- Fraction busy = ρ

where $\rho = \lambda \mathbb{E}\{S\}$, and where $\bar{I}, \bar{V}, \bar{B}$ are as computed above.

b)

$$\bar{W}_q = \frac{\bar{R}}{1 - \rho}$$

where:

$$\bar{R} = \left(\frac{\bar{I}}{\bar{I} + \bar{V} + \bar{B}} \right) \mathbb{E}\{V\} + \left(\frac{\bar{V}}{\bar{I} + \bar{V} + \bar{B}} \right) \frac{\mathbb{E}\{V^2\}}{2\mathbb{E}\{V\}} + \rho \frac{\mathbb{E}\{S^2\}}{2\mathbb{E}\{S\}}$$

where $\bar{V}, \bar{I}, \rho, \bar{B}$ are as computed above. Thus:

$$\bar{W} = \bar{W}_q + \mathbb{E}\{S\}$$

Note: For obvious reasons, it is often easier to grade if you *do not try to simplify your answer*. This makes it easy to see the thought process, and it makes life easier for everyone.

Simplified Answer:

- Fraction idle = $\frac{1 - \rho}{1 + \lambda \mathbb{E}\{V\}}$.
- Fraction set-up = $\frac{\mathbb{E}\{V\}(1 - \rho)\lambda}{1 + \lambda \mathbb{E}\{V\}}$.
- Fraction busy = ρ

Thus:

$$\bar{W}_q = \frac{\lambda \mathbb{E}\{S^2\}}{2(1 - \rho)} + \frac{\mathbb{E}\{V\}}{1 + \lambda \mathbb{E}\{V\}} + \frac{\lambda \mathbb{E}\{V^2\}}{2(1 + \lambda \mathbb{E}\{V\})}$$