EE549: Problem Set #9
Due: Wednesday April 30, 2008

I. RACING EXPONENTIALS

a) Consider a M/M/1 queue with arrival rate λ and server rate μ. Suppose that the system is initially empty. Compute the probability that there are 20 arrivals before the first departure.

b) For the same system as part (a), where the M/M/1 queue is initially empty: compute the probability that there is exactly 1 departure before the first 20 arrivals.

II. TIME TO EMPTY

a) Consider a queue with a buffer and with two homogeneous exponential servers, each of rate μ. Suppose there are 10 packets initially in the system. Thus, there is one packet in each of the two servers, and there are 8 packets in the queue buffer. There are no new arrivals. Compute the expected time until the system empties.

b) Suppose there are two separate queueing systems, each with a single server of rate μ. All packet service times are i.i.d. and exponential of rate μ. There are three packets initially in each system, and there are no arrivals. What is the expected time until both systems are empty? (Note that packets cannot change queues, so that if one system empties before the other, its server remains idle even if the other queue has several packets remaining).

c) Consider a M/M/1/3 queue: Poisson arrivals with rate λ, i.i.d. exponential service times with rate μ, one server, and space for at most 3 packets in total in the system. Suppose that the system initially has 1 packet at time zero. Compute the expected time until the system empties.

III. TRANSIENT ANALYSIS OF A 2-STATE CTMC

Recall the *Kolmogorov Forward Equations* for a finite state CTMC $Z(t)$:

$$\frac{d}{dt}p_j(t) = -v_j p_j(t) + \sum_{i \neq j} p_i(t) q_{ij} \text{ for } j \in \mathcal{S}$$

where $\mathcal{S}$ is the finite state space, and where $p_i(t) = Pr[Z(t) = i]$. Consider a 2-state Markov chain with states 0 and 1. Suppose $q_{01} = \lambda$ and $q_{10} = \mu$. Assume that initial probabilities are given by $p_0(0) = \theta_0$, $p_1(0) = \theta_1$ (where $\theta_0 + \theta_1 = 1$).

a) Write the Kolmogorov differential equation for states $j = 0$ and $j = 1$.

b) Use the fact that $p_0(t) + p_1(t) = 1$ to write the differential equation of part (a) for $j = 0$ purely in terms of the $p_0(t)$ function. The result should be a linear ODE of the form $df(t)/dt = \alpha + \beta f(t)$.

c) Find a *particular* solution of the form of a constant C, so that $p_0(t) = C$ solves the ODE of part (b). (i.e., find a constant C that solves the ODE in part (b)).

d) Find a class of solutions of the form $p_0(t) = Ae^{-\gamma t}$ that solve the *homogeneous version* of the ODE in part (b). (i.e., find γ that solves the homogeneous version of the ODE in part (b). Recall that the homogenous version of an ODE $df(t)/dt = \alpha + \beta f(t)$ is the homogenous ODE $df(t)/dt = \beta f(t)$).

e) Letting $p_0(t) = C + Ae^{-\gamma t}$, solve for the parameters C, A, γ that solve the ODE in part (b) with initial conditions $p_0(t) = \theta_0$ and $p_1(t) = \theta_1$. Give an exact expression for $p_0(t)$ and $p_1(t)$ for all $t \geq 0$.

f) Compute $p_0(\infty)$ and $p_1(\infty)$.

IV. A TOKEN BUFFER SYSTEM WITH POISSON PACKET AND TOKEN ARRIVALS

Consider a token-based admission system where packet arrivals take place according to a Poisson process of rate λ (packets/sec). The packets are stored in a transport layer storage buffer until they can be admitted into the network. Let $L(t)$ represent the number of packets in the storage buffer. Packets are admitted into the network according to a token-based scheme. Specifically, tokens arrive according to a Poisson process of rate γ (tokens/sec), where $\gamma > \lambda$. The tokens are stored in a finite-buffer token bucket with maximum number of tokens given by an integer $T_{max} \geq 1$. Let $T(t)$ represent the number of tokens in the token bucket at time t (where $T(t) \in \{0, 1, \ldots, T_{max}\}$).

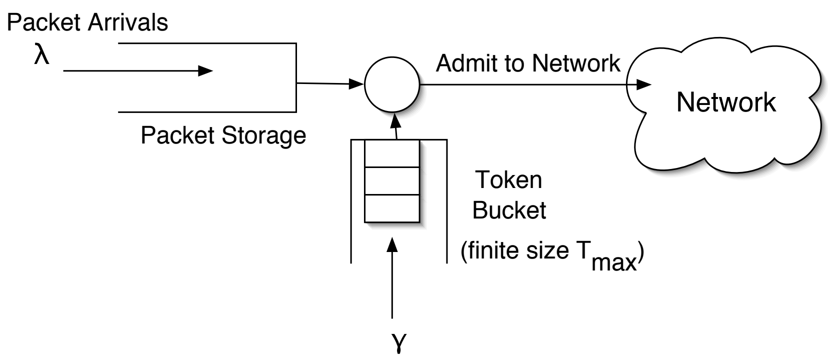

Fig. 1. An illustration of the token-bucket admission system at the transport layer.

Tokens that arrive when $T(t) = T_{max}$ are dropped. A single token can be used for a single packet admission. Thus, if a packet arrives when $T(t) > 1$, a single token is removed from the token bucket, and the packet is admitted to the network. Else, packets must wait in the storage buffer for token arrivals, and the Head-of-Line packet in the storage buffer is admitted whenever we get the next token arrival (and hence uses up the token).

a) Argue that if $L(t) > 0$ then $T(t) = 0$. Likewise, if $T(t) > 0$ then $L(t) = 0$.

b) Let $Z(t) = L(t) - T(t)$. Thus, $Z(t) \in \{-T_{max}, \ldots, -1, 0, 1, 2, 3, \ldots\}$. Draw the Markov chain for $Z(t)$.

c) Compute the expected delay in this token-bucket admission system. That is, compute $\mathbb{E}\{W\}$, where W is the total delay of a packet in the system (the difference between the time the packet is admitted to the network layer and the time when the packet arrives to the transport layer). Note that an individual packet might have a delay of zero if it arrives when $T(t) > 0$.

V. Residual Life

Let $N(t)$ represent a renewal process with i.i.d. inter-renewal times $\{X_i\}$. Suppose the inter-renewal times have a continuous probability density function $p_X(x)$, where $\mathbb{E}\{X\} = \int_0^\infty x p_X(x)dx$ and $\mathbb{E}\{X^2\} = \int_0^\infty x^2 p_X(x)dx$. Recall that the time average residual time to the next renewal converges with probability 1 to $\overline{R} = \frac{\mathbb{E}\{X^2\}}{2\mathbb{E}\{X\}}$.

a) Define $e(t)$ as the *elapsed time* since the last renewal. Draw a picture of a typical $e(t)$ sample path, and compute $\lim_{t\to\infty} \frac{1}{t}\int_0^t e(\tau)d\tau$.

b) Write a sentence summarizing the main result of part (a).

c) Suppose you observe the renewal process at a random time on the timeline. What is the expected size of the renewal interval that you arrive into?

d) For the problem of part (b), let $\tilde{X}$ be the random variable representing the size of the interval you arrive into. Compute the cumulative distribution function $Pr[\tilde{X} \leq x]$ using time average renewal-reward theory. Differentiate to obtain the density, and verify that $\mathbb{E}\left\{\tilde{X}\right\}$ is the same as given in part (b).

Hint: Define the indicator function:

$$1(t) = \begin{cases} 1 & \text{if the renewal interval containing time } t \text{ is less than or equal to } x \text{ in length} \\ 0 & \text{otherwise} \end{cases}$$

VI. M/G/1 Queues

a) Packets arrive according to a Poisson process of rate λ. There are K queues, each with server rate γ kbits/sec. All packets have size B kbits. Assume each packet is independently and uniformly placed into one of the K queues with equal probability $1/K$. Packets cannot change queues once they are routed. Compute the expected delay $\mathbb{E}\{W\}$ in the system (including queueing and service time). Assume that $\lambda < K\gamma/B$.

b) Now suppose that instead of routing to one of K queues, the Poisson process of rate λ (from part (a)) is placed into a single queue with rate $K\gamma$ kbits/sec. Compute $\mathbb{E}\{W\}$ (including queueing and service time), and compare with part (a).

c) Three arrival processes enter a queue with fixed transmission rate γ kbits/sec. Arrival process i has rate λ_i (packets/sec) and all packets have fixed size B_i (kbits), for $i \in \{1, 2, 3\}$. Suppose that $\lambda_1 B_1 + \lambda_2 B_2 + \lambda_3 B_3 < \gamma$.

All packets are served in FIFO order. Compute the delay $\mathbb{E}\{W_q\}$ in the queue buffer (not including service time) for this system. Hint: View the combined stream as a single Poisson arrival process with i.i.d. packet service times.

VII. Queues with Set-Up Time "Vacations"

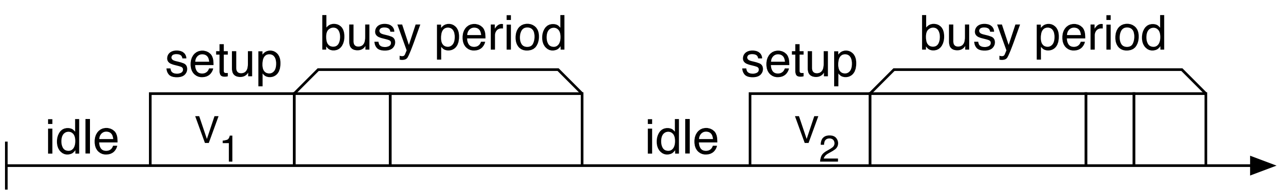

Fig. 2. An example timeline for the queueing system with i.i.d. set-up times.

Consider the following variation of a M/G/1 queue: Packets arrive according to a Poisson process of rate λ. Packets have i.i.d. service times with service time moments $\mathbb{E}\{S\}$ and $\mathbb{E}\{S^2\}$. Packets are served back-to-back FIFO during a busy period. At the end of a busy period, the system is idle. The next packet to arrive during this idle time triggers a "set-up" time before the next busy period (see Fig. 2). Specifically, the busy period cannot start until the set-up time is over. Let V_i represent the set-up time for the ith busy period. The set-up times are i.i.d. with moments $\mathbb{E}\{V\}$ and $\mathbb{E}\{V^2\}$. Assume that $\lambda < 1/\mathbb{E}\{S\}$.

a) Compute the fraction of time the system is in the idle period, the set-up period, and the busy period, respectively.

b) Compute the average delay in the system (including both queueing and service time). Hint: Use the fact that $\overline{W}_q = \overline{R} + \overline{S} \times \overline{N}_q$, where $\overline{R}$ is the residual time until the start of the next service, as seen by an arriving packet. Compute $\overline{R}$ using part (a).

VIII. Ungraded Exercise: Transforms and Decay Factors (You might want to use MATLAB)

This exercise is not graded. Do not turn work in for this problem.

Consider a slotted $GI/D/1$ queue with $a_1 = 1/4, a_2 = 1/8, a_3 = 1/12, a_0 = 1 - a_1 - a_2 - a_3$.

a) Compute the decay factor γ.

b) Give an expression for the single pole approximation $\pi_n \approx R\gamma^n$ (recall that $R = \frac{-(1-\lambda)(1-\gamma)\hat{A}(\gamma)\gamma^{-1}}{\gamma\hat{A}'(\gamma)+\hat{A}(\gamma)}$).

c) Compute the exact probabilities π_n and compare with the single pole approximation.

(Note: Where does the expression for R come from? The constant R is the *residue* of $\hat{\Pi}(z)$ at the pole $z = \gamma$. That is, if we write $\hat{\Pi}(z) = \frac{f(z)}{1-\gamma z^{-1}}$, then $R = f(\gamma)$, and is computed by the following: $R = \lim_{z\to\gamma} \hat{\Pi}(z)(1 - \gamma z^{-1})$.