EE549: Problem Set #7
Due: Monday April 7, 2008

I. k-STEP TRANSITION PROBABILITIES

a)

$$\begin{aligned} Pr[L(1)=0] &= (1-\lambda) \\ Pr[L(1)=1] &= \lambda \\ Pr[L(1)=j] &= 0 \text{ if } j>1 \end{aligned}$$

b)

$$\begin{aligned} Pr[L(2)=0] &= (1-\lambda)^2+\lambda\mu(1-\lambda) \\ Pr[L(2)=1] &= \lambda(\lambda\mu+(1-\lambda)(1-\mu))+(1-\lambda)\lambda \\ Pr[L(2)=2] &= \lambda(1-\mu)\lambda \\ Pr[L(2)=j] &= 0 \text{ if } j>2 \end{aligned}$$

c) With $\lambda=0.4$, $\mu=0.8$ we have:

$$\begin{aligned} Pr[L(1)=0] &= 0.6 \\ Pr[L(1)=1] &= 0.4 \\ Pr[L(1)=j] &= 0 \text{ if } j>1 \end{aligned}$$

$$\begin{aligned} Pr[L(2)=0] &= 0.552 \\ Pr[L(2)=1] &= 0.416 \\ Pr[L(2)=2] &= 0.032 \\ Pr[L(2)=j] &= 0 \text{ if } j>2 \end{aligned}$$

d)

$$Pr[L(3)=0]=Pr[L(2)=0](1-\lambda)+Pr[L(2)=1](1-\lambda)\mu=(0.552)(0.6)+(0.416)(0.48)=0.53088$$

$$\begin{aligned} Pr[L(3)=1] &= Pr[L(2)=0]\lambda+Pr[L(2)=1][\lambda\mu+(1-\lambda)(1-\mu)]+Pr[L(2)=2]\mu(1-\lambda) \\ &= (0.552)(0.4)+(0.416)[(0.4)(0.8)+(0.6)(0.2)]+(0.032)(0.8)(0.6)=0.419200 \end{aligned}$$

$$\begin{aligned} Pr[L(3)=2] &= Pr[L(2)=1]\lambda(1-\mu)+Pr[L(2)=2][\lambda\mu+(1-\lambda)(1-\mu)]+Pr[L(2)=3]\mu(1-\lambda) \\ &= (0.416)(0.4)(0.2)+(0.032)[(0.4)(0.8)+(0.6)(0.2)]=0.047360 \end{aligned}$$

$$Pr[L(3)=3] = Pr[L(2)=2]\lambda(1-\mu)=(0.032)(0.4)(0.2)=0.00256$$

II. A 3-State Markov Chain

a)

$$\begin{aligned} Pr[T_{11} = 1] &= 1/2 \\ Pr[T_{11} = 2] &= (1/2)(3/4) = 3/8 \\ Pr[T_{11} = 3] &= 0 \end{aligned}$$

b) Let $m_i = \mathbb{E}\{\text{time to get back to state 1 given we are in state } i\}$. Thus, $m_1 = \mathbb{E}\{T_{11}\}$. Also:

$$\begin{aligned} m_1 &= 1(1/2) + (1 + m_2)(1/2) \\ m_2 &= 1(3/4) + (1 + m_3)(1/4) \\ m_3 &= 1 + m_2 \end{aligned}$$

Solving yields $m_2 = 5/3 = 1.66666$ and $m_1 = 11/6 = 1.833333$.

c) The chain is clearly irreducible (every state can get to every other one) and is aperiodic because the first state has a self-transition. The detail equations are:

$$\pi_1(1/2) = \pi_2(3/4), \pi_2(1/4) = \pi_3$$

and hence $\pi_2 = \pi_1(2/3)$, $\pi_3 = \pi_1(1/6)$, so:

$$1 = \pi_1(1 + 2/3 + 1/6) = \pi_1(11/6)$$

and therefore $\pi_1 = 6/11$, which corresponds to $1/\mathbb{E}\{T_{11}\}$. Further: $\pi_2 = 4/11$ and $\pi_3 = 1/11$.

III. A 3-state Markov chain and Reverse-Engineering a Queueing Model

a) This finite state irreducible, aperiodic Markov chain must have a steady state distribution. The steady state must satisfy the following cut equations:

$$\begin{aligned} \pi_1(0.2 + 0.3) &= \pi_2(0.5) \\ \pi_3(0.5) &= \pi_1(0.2) + \pi_2(0.2) \end{aligned}$$

and must satisfy $\pi_1 + \pi_2 + \pi_3 = 1$. Therefore:

$$\pi_1 = 0.4166666 \ , \ \pi_2 = 0.4166666 \ , \ \pi_3 = 0.1666666$$

b) This is a discrete time $GI/D/1$ queue with a fixed service rate of 1 packet/slot, a fixed buffer that can hold at most 2 packets, and with i.i.d. arrivals $A(t)$. The queue evolution is given by:

$$L(t+1) = \min[\max[L(t) - 1, 0] + A(t), 2]$$

The distribution of $A(t)$ is given by:

$$Pr[A(t) = 0] = 0.5 \ , \ Pr[A(t) = 1] = 0.3 \ , \ Pr[A(t) = 2] = 0.2$$

IV. A queue with time-correlated ON/OFF channels

a) It is clear that the ON/OFF Markov chain $Z(t)$ has steady state:

$$\pi_{ON} = \frac{\delta}{\epsilon + \delta} \ , \ \pi_{OFF} = \frac{\epsilon}{\epsilon + \delta}$$

Let $N_{ON}(t)$ be the number of times that the Markov chain is ON during $\{0, 1, \ldots, t\}$. By the steady state theorem, we know that $\lim_{t\to\infty} N_{ON}(t) = \delta/(\epsilon + \delta)$ with prob. 1. Therefore:

$$\frac{1}{t}\sum_{\tau=0}^{t-1} A(\tau) = \frac{N_{ON}(t-1)}{t} = \frac{N_{ON}(t-1)}{t-1}\left(\frac{t-1}{t}\right)$$

and hence:

$$\lim_{t\to\infty}\frac{1}{t}\sum_{\tau=0}^{t-1}A(\tau)=\delta/(\epsilon+\delta) \text{ (w.p.1)}$$

b) By Little's Theorem we have that $\rho=\lambda/\mu$, where ρ is the time average fraction of time the system is busy. Thus, $1-\rho=1-\lambda/\mu$ is the fraction of time the system is empty. By ergodicity, we have:

$$\lim_{t\to\infty}Pr[L(t)=0]=1-\rho=1-\lambda/\mu$$

where $\lambda=\delta/(\epsilon+\delta)$.

c) The state space is 2-dimensional and given by (i,ON) and (i,OFF) for $i\in\{0,1,2,\ldots\}$. The "i" represents the current number of packets in the system, and the "ON" or "OFF" represents the current state of $Z(t)$.

d)

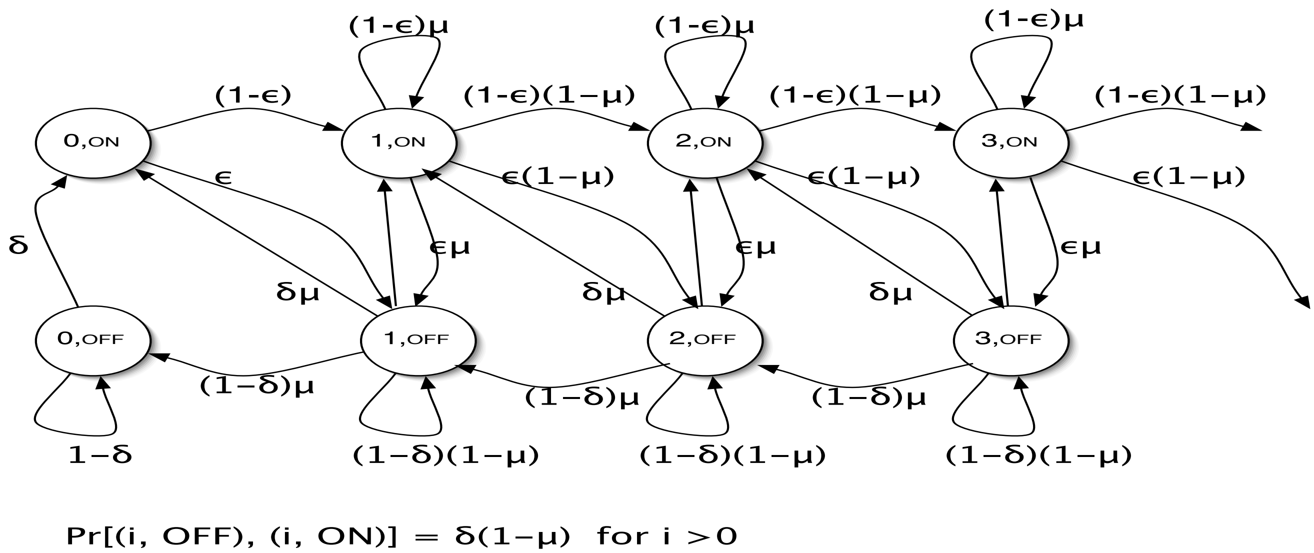

Fig. 1. The 2-dimensional Markov chain.

e) Take the cut that separates states $(0,ON)$ and $(0,OFF)$ from the other states:

$$\pi(0,ON)(1)+\pi(0,OFF)(0)=\pi(1,OFF)\delta\mu$$

V. Opportunistic Scheduling in a 2-user Wireless Downlink with ON/OFF channels

a) Let p_i be the probability that we serve channel $i\in\{1,2\}$. Then:

$$\begin{aligned}p_1&=\phi_1(1-\phi_2)+\phi_1\phi_2\theta_1\\p_2&=\phi_2(1-\phi_1)+\phi_1\phi_2\theta_2\end{aligned}$$

Assuming that $\lambda_1<p_1$ and $\lambda_2<p_2$, each queue is a discrete $GI/B/1$ queue, and so:

$$\begin{aligned}\mathbb{E}\{L_1\}&=\frac{\lambda_1+\mathbb{E}\left\{A_1^2\right\}-2\lambda_1^2}{2(p_1-\lambda_1)}=\frac{B_1}{\alpha\theta_1-\sigma_1}\\\mathbb{E}\{L_2\}&=\frac{\lambda_2+\mathbb{E}\left\{A_2^2\right\}-2\lambda_2^2}{2(p_2-\lambda_2)}=\frac{B_2}{\alpha\theta_2-\sigma_2}\end{aligned}$$

where we have defined new constants $B_1, B_2, \sigma_1, \sigma_2, \alpha$ as follows:

$$\begin{aligned} B_1 &= [\lambda_1 + \mathbb{E}\left\{A_1^2\right\} - 2\lambda_1^2]/2 \\ B_2 &= [\lambda_2 + \mathbb{E}\left\{A_2^2\right\} - 2\lambda_2^2]/2 \\ \alpha &= \phi_1\phi_2 \\ \sigma_1 &= -\phi_1(1-\phi_2) + \lambda_1 \\ \sigma_2 &= -\phi_2(1-\phi_1) + \lambda_2 \end{aligned}$$

Note that Little's theorem says average delay is given by $\overline{W} = \mathbb{E}\left\{L_1 + L_2\right\}/(\lambda_1+\lambda_2)$. Thus, it suffices to minimize $\overline{L}_{tot} = \mathbb{E}\left\{L_1\right\} + \mathbb{E}\left\{L_2\right\}$. Thus, we want to solve:

$$\begin{aligned} \text{Minimize:} \quad & \tfrac{B_1}{\alpha\theta_1 - \sigma_1} + \tfrac{B_2}{\alpha(1-\theta_1)-\sigma_2} \\ \text{Subject to:} \quad & 0 \leq \theta_1 \leq \theta_2 \end{aligned}$$

Taking a derivative with respect to θ_1 and setting the result to zero yields:

$$\frac{B_1\alpha}{(\alpha\theta_1 - \sigma_1)^2} = \frac{B_2\alpha}{(\alpha(1-\theta_1) - \theta_2)^2}$$

and so:

$$\theta_1 = \frac{\frac{\alpha}{\sqrt{B_2}} - \frac{\sigma_2}{\sqrt{B_2}} + \frac{\sigma_1}{\sqrt{B_1}}}{\alpha(\frac{1}{\sqrt{B_1}} + \frac{1}{\sqrt{B_2}})}$$

However, the value of θ_1 needs to satisfy $0 \leq \theta_1 \leq 1$. If the above value is negative, it means we should set $\theta_1 = 0$, and if bigger than 1 it means we should set $\theta_1 = 1$. Thus:

$$\begin{aligned} \theta_1 &= \min\left[\max\left[\frac{\frac{\alpha}{\sqrt{B_2}} - \frac{\sigma_2}{\sqrt{B_2}} + \frac{\sigma_1}{\sqrt{B_1}}}{\alpha(\frac{1}{\sqrt{B_1}} + \frac{1}{\sqrt{B_2}})}, 0\right], 1\right] \\ \theta_2 &= 1 - \theta_1 \end{aligned}$$

b) For these parameters we have:

$$B_1 = 0.21, B_2 = 0.16, \alpha = 0.27, \sigma_1 = -0.33, \sigma_2 = 0.17$$

and hence:

$$\theta_1 = 0, \theta_2 = 1$$

So:

$$\overline{L}_{tot} = 2.236364, \overline{W} = 3.727273$$

c) For these parameters we have:

$$B_1 = 0.21, B_2 = 0.16, \alpha = 0.72, \sigma_1 = 0.12, \sigma_2 = 0.12$$

and hence:

$$\theta_1 = 0.522626, \theta_2 = 0.477374$$

So:

$$\overline{L}_{tot} = 1.534596, \overline{W} = 2.557660$$

d) Server policy: Every slot t, observe the set of ON queues, and independently choose an ON queue to serve, uniformly over all ON queues.

Analysis: Let $p_1, \ldots, p_K$ represent the the probability that we serve queues 1 to K, respectively, so that: $p_i = Pr[\text{serve queue } i \text{ on slot } t]$. By the symmetry of the problem, we have:

$$p_1 = p_2 = \ldots = p_K$$

Now define μ as the probability that at least one channel is ON:

$$\mu \triangleq 1 - (1-\phi)^K$$

Because the above server policy always serves a queue whenever there is at least one ON channel, we have:

$$p_1 + p_2 + \ldots + p_K = \mu$$

Therefore, for all $i \in \{1, \ldots, K\}$ we have:

$$p_i = \frac{\mu}{K}$$

Because $\lambda_{tot} < \mu$, we know that $\lambda_i = \lambda_{tot}/K < \mu/K = p_i$. It follows that for all $i \in \{1, 2, \ldots, K\}$, we have:

$$\mathbb{E}\{L_i\} = \frac{\lambda_i + \mathbb{E}\{A_i^2\} - 2\lambda_i^2}{2(p_i - \lambda_i)} = \frac{\lambda_{tot} + K\mathbb{E}\{A^2\} - 2\lambda_{tot}^2/K}{2(\mu - \lambda_{tot})}$$

Thus:

$$\mathbb{E}\{L_1 + \ldots L_K\} = \frac{K\lambda_{tot} + K^2\mathbb{E}\{A^2\} - 2\lambda_{tot}^2}{2(\mu - \lambda_{tot})}$$

By Little's theorem, the average delay is:

$$\overline{W} = \frac{K + \frac{K^2}{\lambda_{tot}}\mathbb{E}\{A^2\} - 2\lambda_{tot}}{2(\mu - \lambda_{tot})}$$

For Bernoulli traffic we have $\mathbb{E}\{A^2\} = \mathbb{E}\{A\} = \lambda_{tot}/K$, and so average delay satisfies:

$$\overline{W}_{Bernoulli} = \frac{K - \lambda_{tot}}{\mu - \lambda_{tot}}$$

VI. Periodic Service

a) We have:

$$\mathbb{E}\{L(2k+2)\} = \mathbb{E}\{L(2k+1)\} + \mathbb{E}\{A(2k+1)\} = \mathbb{E}\{L(2k+1)\} + \lambda$$

and hence, taking limits as $k \to \infty$:

$$\mathbb{E}\{L \,|\, even\} = \mathbb{E}\{L \,|\, odd\} + \lambda$$

Thus:

$$\overline{L} = \frac{1}{2}\mathbb{E}\{L \,|\, even\} + \frac{1}{2}[\mathbb{E}\{L \,|\, even\} - \lambda] = \mathbb{E}\{L \,|\, even\} - \lambda/2$$

b)

$$\begin{aligned} L(2k+2) &= L(2k+1) + A(2k+1) \\ &= \max[L(2k) - 1, 0] + A(2k) + A(2k+1) \end{aligned}$$

Let $\hat{A}(2k) = A(2k) + A(2k+1)$. Note that $\mathbb{E}\left\{\hat{A}(2k)\right\} = 2\lambda$ and $\mathbb{E}\{A(2k)^2\} = 2\lambda + 2\lambda^2$. Thus:

$$L(2k+2) = L(2k) - \tilde{\mu}(2k) + \hat{A}(2k) \quad (1)$$

where $\tilde{\mu}(2k) = 1_{L(2k)>0}$. Taking expectations of (1) (and assuming the steady state expectations exist) yields:

$$\mathbb{E}\{L(2k+2)\} = \mathbb{E}\{L(2k)\} - \mathbb{E}\{\tilde{\mu}(2k)\} + 2\lambda$$

Thus:

$$\lim_{k\to\infty} \mathbb{E}\{\tilde{\mu}(2k)\} = 2\lambda$$

Squaring (1) yields:

$$L(2k+2)^2 = L(2k)^2 + \tilde{\mu}(2k) - 2\tilde{\mu}(2k)A(2k) + \hat{A}(2k)^2 - 2L(2k)[1 - \hat{A}(2k)]$$

where we have used the fact that $\tilde{\mu}(2k)^2 = \tilde{\mu}(2k)$ (because $\tilde{\mu}(2k) \in \{0, 1\}$), and $L(2k)\tilde{\mu}(2k) = L(2k)$. Taking expectations and limits yields:

$$0 = 2\lambda - 2(2\lambda)(2\lambda) + 2\lambda + 2\lambda^2 - 2\mathbb{E}\{L \,|\, even\}(1 - 2\lambda)$$

Therefore:

$$\mathbb{E}\{L \mid even\} = \frac{2\lambda - 3\lambda^2}{(1-2\lambda)}$$

Hence, from part (b) we have:

$$\overline{L} = \frac{2\lambda - 3\lambda^2}{(1-2\lambda)} - \lambda/2$$

d) The average delay is given by Little's theorem:

$$\overline{W} = \frac{2 - 3\lambda}{(1-2\lambda)} - 1/2$$

For $\lambda = 1/4$ we have:

$$\overline{W} = \frac{2 - 3/4}{1 - 1/2} = 4 - 1.5 - 0.5 = 2.0 \text{ slots}$$

A $B/B/1$ queue with $\lambda = 1/4$ and $\mu = 1/2$ has average delay:

$$\overline{W}_{B/B/1} = \frac{1-\lambda}{\mu - \lambda} = \frac{3/4}{1/4} = 3.0 \text{ slots}$$

Thus, the periodic system with average service rate $1/2$ gives a better average delay than the randomized service system with average service rate $1/2$.

VII. A 7-State Markov Chain

a) $k_0 = 8$. Proof: We can have a cycle from 1 to 1 of length h, for any h such that:

- $h \mod 3 = 0$ (because $h = 3m$ for some positive integer m).
- $h \mod 3 = 2$ and $h \geq 5$ (because $h = 5 + 3m$ for some non-negative integer m).
- $h \mod 3 = 1$ and $h \geq 10$ (because $h = 5 + 5 + 3m$ for some non-negative integer m).

Considering the above, we see we can achieve any integer h such that $h \geq 10$. Now observe that it is also possible to achieve $h \in \{8, 9\}$, but it is not possible to achieve $h = 7$. Therefore, $k_0 = 8$.

b) $M = 12$. For any state that is z hops away from state 0, it is possible to find a path from the state to 0 that uses $z + k$ hops, for any $k \geq k_0$. The state furthest away from state 0 is 1 (requiring 4 hops). Thus, $M = k_0 + 4 = 8 + 4 = 12$.

c)

$$P = \begin{bmatrix} 0 & 1/2 & 0 & 0 & 0 & 1/2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

d) We have from the GBEs:

$$\pi_2 = \pi_1(1), \pi_3 = \pi_2(1), \pi_4 = \pi_3(1), \pi_6 = \pi_5(1)$$

Thus:

$$\pi_1 = \pi_2 = \pi_3 = \pi_4, \pi_5 = \pi_6$$

Further:

$$\pi_1 = \pi_0(1/2), \pi_5 = \pi_0(1/2)$$

Thus:

$$4\pi_0(1/2) + \pi_0 + 2\pi_0(1/2) = 1$$

Hence:

$$\pi_0 = \frac{1}{2+1+1} = 1/4$$

$$\pi_1 = \pi_2 = \pi_3 = \pi_4 = 1/8$$

$$\pi_5 = \pi_6 = 1/8$$

e) The initial probability distribution is $\vec{\pi}(0) = (1, 0, 0, 0, 0, 0, 0)$. Then: $\vec{\pi}(t+1) = \vec{\pi}(t)P$ and hence $\vec{\pi}(t) = \vec{\pi}(0)P^t$ for $t \in \{0, 1, 2, \ldots\}$. The graph should show that all probabilities for state $i \neq 0$ quickly converge to $1/8 = 0.125$, while $\pi_0(t)$ quickly converges to $1/4 = 0.25$.