EE549: Problem Set #7
Due: Monday April 7, 2008

I. k-STEP TRANSITION PROBABILITIES

Let $L(t)$ represent the number of packets in a discrete time $B/B/1$ queue with arrival rate λ and server rate μ. Suppose that $L(0) = 0$.

a) Compute $Pr[L(1) = j]$ for all $j \in \{0, 1, 2, \ldots\}$.

b) Compute $Pr[L(2) = j]$ for all $j \in \{0, 1, 2, \ldots\}$.

c) Give numerical answers for (a) and (b) for $\lambda = 0.4$, $\mu = 0.8$.

d) Give a numerical answer for $Pr[L(3) = j]$ for all $j \in \{0, 1, 2, \ldots\}$ (for the case $\lambda = 0.4$, $\mu = 0.8$).

II. A 3-STATE MARKOV CHAIN

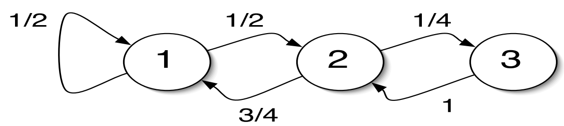


Fig. 1. A 3-state Markov chain.

Consider the 3-state Markov chain $Z(t)$ of Fig. 1.

a) Suppose that $Z(0) = 1$. Let T_{11} be the random time until we reach state 1 again (so that $T_{11} \in \{1, 2, 3, \ldots\}$). Compute $Pr[T_{11} = 1]$. Compute $Pr[T_{11} = 2]$. Compute $Pr[T_{11} = 3]$.

b) Compute $\mathbb{E}\{T_{11}\}$.

c) Is this chain irreducible and aperiodic? Use the GBE equations (or the detail equations) to solve for the steady state distribution π_1, π_2, π_3. Show that $\pi_1 = 1/\mathbb{E}\{T_{11}\}$.

III. A 3-STATE MARKOV CHAIN AND REVERSE-ENGINEERING A QUEUEING MODEL

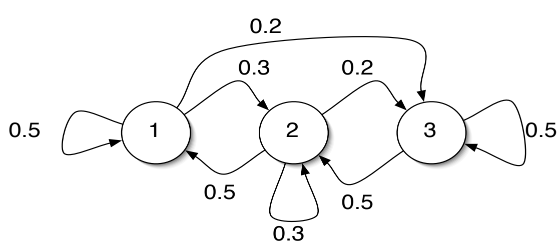


Fig. 2. A 3-state Markov chain.

Consider the 3-state Markov chain with transition probabilities given in Fig. 2. Note that the chain is irreducible and aperiodic.

a) Solve for the steady state probabilities π_1, π_2, π_3.

b) Define a queueing model that corresponds to this Markov chain.

IV. A QUEUE WITH TIME-CORRELATED ON/OFF CHANNELS

Let $L(t)$ represent the number of packets in a discrete time queue. Assume all packet sizes are fixed, and the queue evolves according to the following equation:

$$L(t+1) = \max[L(t) - \mu(t), 0] + A(t)$$

Assume that the service rate process $\mu(t)$ is i.i.d. Bernoulli with $Pr[\mu(t) = 1] = \mu$. However, assume that arrivals $A(t)$ are *time-correlated* in the following way: Let $Z(t)$ be a 2-state discrete time Markov chain with states ON and OFF, as shown in Fig. 3.

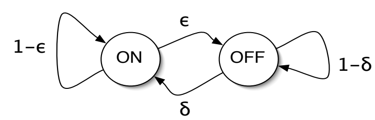


Fig. 3. A 2-state ON/OFF Markov chain for the arrival process $A(t)$.

The arrival process $A(t)$ is determined from $Z(t)$ as follows:

$$A(t) = \begin{cases} 1 & \text{if } Z(t) = ON \\ 0 & \text{if } Z(t) = OFF \end{cases}$$

a) Show that there is a constant λ such that:

$$\lim_{t\rightarrow\infty} \frac{1}{t}\sum_{\tau=0}^{t-1} A(\tau) = \lambda \text{ with prob. } 1$$

Compute λ.

b) Assume that $\lambda < \mu$. Compute $\lim_{t\rightarrow\infty} Pr[L(t) = 0]$. (Hint: Use Little's Theorem).

c) Define a state space that describes this queueing system. Recall that a state should represent all information relevant about the system that is required to simulate the system for future slots. The state space should allow the Markov property, so that the future is conditionally independent of the past given the current state.

d) Draw the Markov chain for this system, using your state space. (Hint: This is *not* a birth-death chain).

e) Write a cut-set equation for a cut in your chain that separates the states and has at least two states on either side of the cut.

V. OPPORTUNISTIC SCHEDULING IN A 2-USER WIRELESS DOWNLINK WITH ON/OFF CHANNELS

Consider a 2-user wireless downlink with 2 queues that operate in discrete time. The arrival processes to each queue are independent and given by $A_1(t)$ and $A_2(t)$, representing the number of packets that arrive to queue 1 and queue 2, respectively, on slot t. All packets have a fixed length. Arrivals from $A_1(t)$ are i.i.d. over slots with mean $\mathbb{E}\{A_1(t)\} = \lambda_1$ and second moment $\mathbb{E}\left\{A_1(t)^2\right\} = \mathbb{E}\left\{A_1^2\right\}$. Likewise, arrivals from $A_2(t)$ are i.i.d. over slots with mean and second moment λ_2 and $\mathbb{E}\left\{A_2^2\right\}$. Channel 1 is an i.i.d. Bernoulli ON/OFF process with ON probability ϕ_1. Channel 2 is independent of channel 1, and is an i.i.d. Bernoulli ON/OFF process with ON probability ϕ_2. There is a single server that can serve exactly one packet during a slot, and can serve a packet only if the corresponding channel is ON.

Assume that $\lambda_1 < \phi_1$, $\lambda_2 < \phi_2$, and $\lambda_1 + \lambda_2 < \phi_1 + (1 - \phi_1)\phi_2$, so that it is possible to stabilize the system.

a) Suppose we use a probabilistic server scheduling policy that bases scheduling decisions purely on the channel state (and independent of queue backlog). In the case when both channels are ON, the server chooses queue 1 with probability θ_1 and chooses queue 2 with probability θ_2 (where $\theta_1 + \theta_2 = 1$). Design parameters θ_1 and θ_2 that minimize average delay in the system. Your answer should be in terms of the general parameters $\phi_1, \phi_2, \lambda_1, \lambda_2, \mathbb{E}\left\{A_1^2\right\}, \mathbb{E}\left\{A_2^2\right\}$, or constants that are defined by these parameters.

b) What is the resulting average delay in your system for $\mathbb{E}\left\{A_1^2\right\} = \lambda_1 = 0.3$, $\mathbb{E}\left\{A_2^2\right\} = \lambda_2 = 0.2$, and $\phi_1 = 0.9, \phi_2 = 0.3$?

c) What is the resulting average delay when $\mathbb{E}\left\{A_1^2\right\} = \lambda_1 = 0.3$, $\mathbb{E}\left\{A_2^2\right\} = \lambda_2 = 0.2$, and $\phi_1 = 0.9, \phi_2 = 0.8$?

d) Suppose we have a K-queue downlink with ON/OFF channels that are symmetric. Specifically, each channel process is an independent i.i.d. Bernoulli ON/OFF process with the same ON probability ϕ. Likewise, suppose that all arrival processes $A_i(t)$ are independent and i.i.d. over slots with the same mean $\lambda_i = \lambda_{tot}/K$ and the same second moment $\mathbb{E}\left\{A_i^2\right\} = \mathbb{E}\left\{A^2\right\}$ for all $i \in \{1, 2, \ldots, K\}$. Suppose that $\lambda_{tot} < 1 - (1-\phi)^K$. Again there is only one server that can serve at most one ON channel per slot. Design a scheduling policy that stabilizes the system, and compute an exact expression for average delay in the system. (Hint: Use the symmetry of the problem). What is the average delay in the special case of symmetric Bernoulli packet arrivals?

VI. Periodic Service

Recall that the average number of packets in a discrete time $GI/B/1$ queue satisfies:

$$\mathbb{E}\{L\} = \overline{L} = \frac{\mathbb{E}\{A\} + \mathbb{E}\left\{A^2\right\} - 2\mathbb{E}\{A\}^2}{2(\mu - \mathbb{E}\{A\})}$$

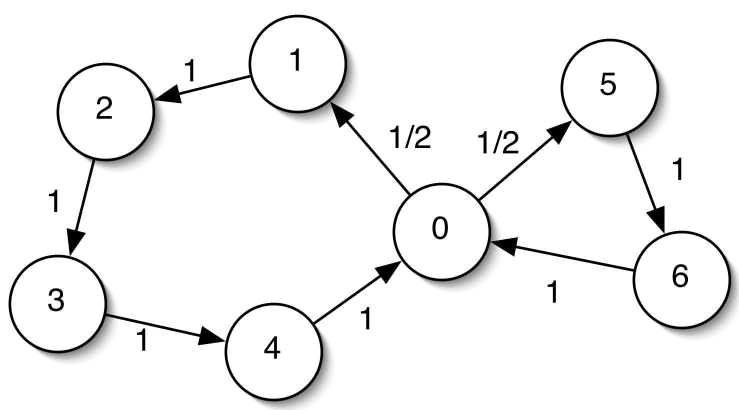
However, rather than *Bernoulli* service, in this problem we consider *periodic* service: Suppose that arrivals $A(t)$ are *i.i.d. Bernoulli with rate* λ, where $\lambda < 1/2$. A service opportunity arises every even timeslot (where a single packet can be served). No service occurs on odd slots. Thus, for any integer k we have the following dynamic queueing equations:

- Odd Slots: $L(2k+1) = \max[L(2k) - 1, 0] + A(2k)$
- Even Slots: $L(2k+2) = L(2k+1) + A(2k+1)$

Note that $\overline{L} = \frac{1}{2}\mathbb{E}\{L \mid even\} + \frac{1}{2}\mathbb{E}\{L \mid odd\}$, where $\mathbb{E}\{L \mid even\}$ and $\mathbb{E}\{L \mid odd\}$ represent the steady state conditional average number of packets in the queue, given the timeslot is even and given the timeslot is odd, respectively.

a) Suppose that $\mathbb{E}\{L \mid even\}$ is known. Give an expression for $\overline{L}$ in terms of $\mathbb{E}\{L \mid even\}$.
b) Write a dynamic queueing equation for $L(2k+2)$ in terms of only $L(2k)$ and the arrival process.
c) Compute $\overline{L}$ in terms of λ (recall that arrivals $A(t)$ are i.i.d. Bernoulli with rate λ).
d) What is the average delay in the system, assuming that $\lambda = 1/4$? Is this periodic service better or worse than a $B/B/1$ queue with $\mu = 1/2$?

VII. A 7-State Markov Chain



Consider the 7-state Markov chain in Fig. VII.

a) Find the smallest integer k_0 such that the k-step transition probability for the cycle $0 \to 0$ satisfies: $P_{00}^{(k)} > 0$ for all $k \geq k_0$. (Hence, the chain is *aperiodic*). You should prove that all $k \geq k_0$ are possible.

b) Find an integer M such that the M-step transition probability for reaching state 0 starting in state i satisfies: $P_{i0}^{(M)} > 0$ for all states $i \in \{0, 1, 2, 3, 4, 5, 6\}$. (Hint: Use the fact that an integer k_0 exists with the properties given in part (a)). Thus, it is possible to get to state 0 from any state in exactly M steps.

c) Find the probability transition matrix $P = (P_{ij})$ (please keep the order of states $\{0, 1, 2, 3, 4, 5, 6\}$, so that everyone has the same matrix).

d) Compute the stationary probability $\vec{\pi}$.

e) Assuming we are initially in state 0, use a computer to compute $\vec{\pi}(t)$ for $t \in \{0, \ldots, 70\}$. Plot the six probabilities of the $\vec{\pi}(t)$ vector versus t, and include the horizontal asymptotes obtained from part (b). (MATLAB is best for this type of computation).

UNGRADED EXTENSION TO PROBLEM VII – MEETINGS OF INDEPENDENT MARKOV RANDOM WALKS

(This problem is not graded and should not be turned in.)

Consider the Markov chain of the previous problem.

i) Define $q \triangleq \min_{i \in \{0,\ldots,6\}} P_{i0}^{(k)}$. Show that $q > 0$, and that for any time t, the probability of the event $\{L(t+k) = \tilde{L}(t+k) = 0\}$ is at least q^2, independently of the state of either Markov process at time t. (where k is the fixed value computed in part (b)).

ii) Let K be the first timeslot in which $L(t)$ and $\tilde{L}(t)$ meet (if they never meet, then define $K = \infty$). Show that $Pr[K > t] \rightarrow 0$ as $t \rightarrow \infty$, and so the random variable K is finite with probability 1. (Actually, you will find that this probability decreases *geometrically*).