EE549: Problem Set #4 part 2 Solutions

This is part 2 of the problem set 4, and is worth 50 points.

I. Integrating a Periodic Function

First observe that any periodic function of period P satisfies:

$$\mu(t - kP) = \mu(t) \quad \text{for any integer } k \text{ and any time } t \tag{1}$$

Now fix any time t. Note that t can be written $t = kP + r$, where k is an integer and r is a real number that satisfies $0 \leq r < P$. Then we have $t < (k+1)P \leq t + P$. Thus:

$$\begin{aligned}
\int_t^{t+P} \mu(\tau)d\tau &= \int_t^{(k+1)P} \mu(\tau)d\tau + \int_{(k+1)P}^{t+P} \mu(\tau)d\tau \\
&= \int_{kP+r}^{(k+1)P} \mu(\tau)d\tau + \int_{(k+1)P}^{(k+1)P+r} \mu(\tau)d\tau && (2)\\
&= \int_{kP+r}^{(k+1)P} \mu(\tau - kP)d\tau + \int_{(k+1)P}^{(k+1)P+r} \mu(\tau - (k+1)P)d\tau && (3)\\
&= \int_r^P \mu(\tau)d\tau + \int_0^r \mu(\tau)d\tau && (4)\\
&= \int_0^P \mu(\tau)d\tau = A && (5)
\end{aligned}$$

where (2) follows because $t = kP + r$, (3) follows because of (1), and (4) follows from a change of variables.

II. Leaky Bucket Inputs into Trees

a) The first 3 queues are easy:

$$\begin{aligned}
U_1(t) &\leq \sigma 1 \\
U_2(t) &\leq \sigma_2 \\
U_3(t) &\leq \sigma_3
\end{aligned}$$

Let $Y_1(t), Y_2(t), Y_3(t)$ be the departure process of queues 1, 2, 3, respectively. Then $Y_i(t) \sim (r_i, \sigma_i)$ for $i \in \{1, 2, 3\}$ by the I-O invariance property (note that $r_i \leq \mu_i$ for $i \in \{1, 2, 3\}$). Thus, $Y_1(t) + Y_2(t) + Y_3(t) \sim (r_1 + r_2 + r_3, \sigma_1 + \sigma_2 + \sigma_3)$, and so (because $r_1 + r_2 + r_3 \leq \mu_4$):

$$U_4(t) \leq \sigma_1 + \sigma_2 + \sigma_3$$

Let $Y_4(t)$ be the output process of queue 4. Then $Y_4(t) \sim (r_1 + r_2 + r_3, \sigma_1 + \sigma_2 + \sigma_3)$. This $Y_4(t)$ combines with the (r_5, σ_5) process as inputs to node 5, and thus (because $r_1 + r_2 + r_3 + r_5 \leq \mu_5$):

$$U_5(t) \leq \sigma_1 + \sigma_2 + \sigma_3 + \sigma_5$$

b) We have already shown above that $Y_4(t) \sim (r_1 + r_2 + r_3, \sigma_1 + \sigma_2 + \sigma_3)$.

c) We have:

$$\begin{aligned}
\text{worst case end-to-end delay of stream 1} &\leq Delay_1 + Delay_4 + Delay_5 \\
&\leq \sigma_1/\mu_1 + (\sigma_1 + \sigma_2 + \sigma_3)/\mu_4 + (\sigma_1 + \sigma_2 + \sigma_3 + \sigma_5)/\mu_5
\end{aligned}$$

d) As before, we have:

$$U_1(t) \leq \sigma_1$$
$$U_2(t) \leq \sigma_2$$
$$U_3(t) \leq \sigma_3$$

Let $\tilde{Y}_1(t), \tilde{Y}_2(t), \tilde{Y}_3(t)$ be the packet-based departure process of queues 1, 2, 3, respectively. Then $\tilde{Y}_i(t) \sim (r_i, \sigma_i + B_{max})$ for $i \in \{1, 2, 3\}$ by the I-O property for packet-based departures (note that $r_i \leq \mu_i$ for $i \in \{1, 2, 3\}$). Thus, $\tilde{Y}_1(t) + \tilde{Y}_2(t) + \tilde{Y}_3(t) \sim (r_1 + r_2 + r_3, \sigma_1 + \sigma_2 + \sigma_3 + 3B_{max})$, and so (because $r_1 + r_2 + r_3 \leq \mu_4$):

$$U_4(t) \leq \sigma_1 + \sigma_2 + \sigma_3 + 3B_{max}$$

Let $\tilde{Y}_4(t)$ be the output process of queue 4. Then $\tilde{Y}_4(t) \sim (r_1+r_2+r_3, \sigma_1+\sigma_2+\sigma_3+4B_{max})$. This $Y_4(t)$ combines with the (r_5, σ_5) process as inputs to node 5, and thus (because $r_1 + r_2 + r_3 + r_5 \leq \mu_5$):

$$U_5(t) \leq \sigma_1 + \sigma_2 + \sigma_3 + \sigma_5 + 4B_{max}$$

We note that we have shown:

$$\tilde{Y}_4(t) \sim (r_1 + r_2 + r_3, \sigma_1 + \sigma_2 + \sigma_3 + 4B_{max})$$

Finally:

$$\begin{aligned} \text{worst case end-to-end delay of stream 1} &\leq Delay_1 + Delay_4 + Delay_5 \\ &\leq \sigma_1/\mu_1 + (\sigma_1 + \sigma_2 + \sigma_3 + 3B_{max})/\mu_4 \\ &\quad +(\sigma_1 + \sigma_2 + \sigma_3 + \sigma_5 + 4B_{max})/\mu_5 \end{aligned}$$

e) The answers of (a)-(c) are the same under the packet-based departure model with constant size packets B, as the I-O invariance property holds exactly as before in this case (so that B_{max} does not need to be added).

III. Network Calculus with Space Networks

a) We draw the queueing model of the space network in Fig. 1.

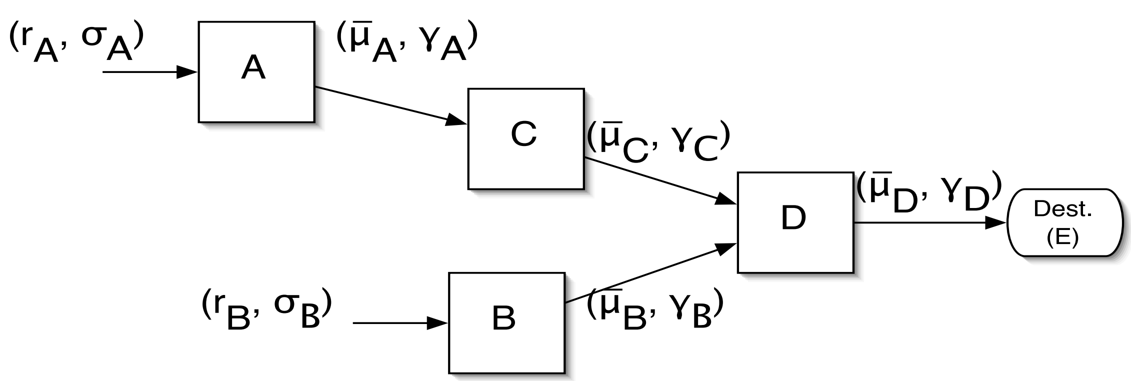

Fig. 1. The queueing model of the space network.

We have:

$$\begin{aligned} \overline{\mu}_A &= x_{AC}(1/5) \text{ bits/sec} \quad , \quad \gamma_A = (3600)4\overline{\mu}_A \text{ bits} \\ \overline{\mu}_B &= x_{BD}(2/15) \text{ bits/sec} \quad , \quad \gamma_B = (3600)13\overline{\mu}_B \text{ bits} \\ \overline{\mu}_C &= x_{CD}(1/24) \text{ bits/sec} \quad , \quad \gamma_C = (3600)23\overline{\mu}_C \text{ bits} \\ \overline{\mu}_D &= x_{DE}(1/12) \text{ bits/sec} \quad , \quad \gamma_D = (3600)11\overline{\mu}_D \text{ bits} \end{aligned}$$

Specifically:

$$\begin{aligned}\gamma_A &= (3600)x_{AC}(4/5) \text{ bits}\\ \gamma_B &= (3600)x_{BD}(26/15) \text{ bits}\\ \gamma_C &= (3600)x_{DC}(23/24) \text{ bits}\\ \gamma_D &= (3600)x_{DE}(11/12) \text{ bits}\end{aligned}$$

b) Constraints that act as a sufficient condition for stability:

$$\begin{aligned}r_A &\leq \overline{\mu}_A\\ r_A &\leq \overline{\mu}_C\\ r_B &\leq \overline{\mu}_B\\ r_A + r_B &\leq \overline{\mu}_D\end{aligned}$$

where the $\overline{\mu}_A, \overline{\mu}_B, \overline{\mu}_C, \overline{\mu}_D$ parameters are given in part (a).

c) Assuming the constraints of part (b) are satisfied, we have:

$$\begin{aligned}U_A(t) &\leq \sigma_A + \gamma_A\\ U_B(t) &\leq \sigma_B + \gamma_B\\ U_C(t) &\leq \sigma_A + \gamma_A + \gamma_C\\ U_D(t) &\leq (\sigma_A + \gamma_A + \gamma_C) + (\gamma_B + \sigma_B) + \gamma_D\end{aligned}$$

where the $\gamma_A, \gamma_B, \gamma_C, \gamma_D$ parameters are given in part (a).

d) The worst case delay in each node is given by:

$$Delay_A \leq \frac{\sigma_A + \gamma_A}{\overline{\mu}_A} \quad (6)$$

$$Delay_B \leq \frac{\sigma_B + \gamma_B}{\overline{\mu}_B} \quad (7)$$

$$Delay_C \leq \frac{\sigma_A + \gamma_A + \gamma_C}{\overline{\mu}_C} \quad (8)$$

$$Delay_D \leq \frac{\sigma_A + \gamma_A + \gamma_C + \sigma_B + \gamma_B + \gamma_D}{\overline{\mu}_D} \quad (9)$$

The worst case end-to-end delay for packets from planet A to reach the destination planet E is thus:

$$\text{worst case delay for } A \leq Delay_A + Delay_C + Delay_D$$

where bounds for $Delay_A, Delay_C, Delay_D$ are given in (6)-(9), and where the $\overline{\mu}_A, \overline{\mu}_B, \overline{\mu}_C, \overline{\mu}_D$ and $\gamma_A, \gamma_B, \gamma_C, \gamma_D$ parameters are given in part (a).

IV. Multi-Input Multi-Output Feedforward Networks

a) The combined process into node A is leaky bucket with parameters $(r_1 + r_2 + r_3, \sigma_1 + \sigma_2 + \sigma_3)$, and so (because $r_1 + r_2 + r_3 \leq \mu_A$:

$$U_A(t) \leq \sigma_1 + \sigma_2 + \sigma_3$$

Let $Y_{AB}(t)$ represent the combined output process of node A that is going to node B (consisting of data from streams 1 and 2), and let $Y_{AC}(t)$ represent the data departing from A that is going to node C (consisting of data from only stream 3). Let $X_{AB}(t) = X_1(t) + X_2(t)$ represent the combined input to node A for this data. Then $X_{AB}(t)$ data has parameters $(r_1 + r_2, \sigma_1 + \sigma_2)$, and mixes with the $X_3(t)$ data, where $X_3(t) \sim (r_3, \sigma_3)$. From Fact 4, we thus have:

$$\begin{aligned}Y_{AB}(t) &\sim (r_1 + r_2, \sigma_1 + \sigma_2 + \sigma_3)\\ Y_{AC}(T) &\sim (r_3, \sigma_1 + \sigma_2 + \sigma_3)\end{aligned}$$

It follows that (because $r_1 + r_2 \leq \mu_B$ and $r_3 + r_4 \leq \mu_C$:

$$U_B(t) \leq \sigma_1 + \sigma_2 + \sigma_3$$
$$U_C(t) \leq \sigma_1 + \sigma_2 + \sigma_3 + \sigma_4$$

We now find parameters for the output processes $Y_{CD}(t)$ from node C to node D (consisting of stream 3 and 4 data) and for $Y_{BD}(t)$ from node B to node D (consisting of only stream 2 data). The combined input to node C has parameters $(r_3 + r_4, \sigma_1 + \sigma_2 + \sigma_3 + \sigma_4)$, and so by I-O invariance:

$$Y_{CD}(t) \sim (r_3 + r_4, \sigma_1 + \sigma_2 + \sigma_3 + \sigma_4)$$

The process $Y_{BD}(t)$ is more tricky. Note that node B has two arrival streams, namely, $Y_{AB}^{(1)}(t)$ and $Y_{AB}^{(2)}(t)$, being the arrivals from node A due to type 1 and 2 data, respectively. $Y_{AB}^{(1)}(t)$ can be viewed as the output of stream 1 in node A due to the (r_1, σ_1) stream 1 packets mixing with the $(r_2 + r_3, \sigma_2 + \sigma_3)$ packets, and so by Fact 4 we have:

$$Y_{AB}^{(1)}(t) \sim (r_1, \sigma_1 + \sigma_2 + \sigma_3)$$

Similarly, we have:

$$Y_{AB}^{(2)}(t) \sim (r_2, \sigma_1 + \sigma_2 + \sigma_3)$$

The $Y_{BD}(t)$ process can be viewed as the output of node B when the input $Y_{AB}^{(2)}(t)$ mixes with $Y_{AB}^{(1)}(t)$. Thus, again by Fact 4, we have:

$$Y_{BD}(t) \sim (r_2, 2\sigma_1 + 2\sigma_2 + 2\sigma_3)$$

Thus, at node D we have a $(r_2, 2\sigma_1 + 2\sigma_2 + 2\sigma_3)$ input mixing with a $(r_3 + r_4, \sigma_1 + \sigma_2 + \sigma_3 + \sigma_4)$ input, and so (because $r_2 + r_3 + r_4 \leq \mu_D$):

$$U_D(t) \leq 3(\sigma_1 + \sigma_2 + \sigma_3) + \sigma_4$$

b) Let us first compute the worst case bit delays in queues A, B, C, D:

$$\begin{aligned}
Delay_A &\leq U_A^{max}/\mu_A = (\sigma_1 + \sigma_2 + \sigma_3)/\mu_A \triangleq Delay_A^{max} \\
Delay_B &\leq U_B^{max}/\mu_B = (\sigma_1 + \sigma_2 + \sigma_3)/\mu_B \triangleq Delay_B^{max} \\
Delay_C &\leq U_C^{max}/\mu_C = (\sigma_1 + \sigma_2 + \sigma_3 + \sigma_4)/\mu_C \triangleq Delay_C^{max} \\
Delay_D &\leq U_D^{max}/\mu_D = (3(\sigma_1 + \sigma_2 + \sigma_3) + \sigma_4)/\mu_D \triangleq Delay_D^{max}
\end{aligned}$$

Then:

$$\begin{aligned}
\text{Worst case end-to-end Delay of Stream 1} &\leq Delay_A^{max} + Delay_B^{max} \\
\text{Worst case end-to-end Delay of Stream 2} &\leq Delay_A^{max} + Delay_B^{max} + Delay_D^{max} \\
\text{Worst case end-to-end Delay of Stream 3} &\leq Delay_A^{max} + Delay_C^{max} + Delay_D^{max} \\
\text{Worst case end-to-end Delay of Stream 4} &\leq Delay_C^{max} + Delay_D^{max}
\end{aligned}$$

where $Delay_A^{max}, Delay_B^{max}, Delay_C^{max}, Delay_D^{max}$ are as given above.