

EE549: Problem Set #3

Solutions

I. OH, THE TROUBLES I'VE SEEN

a) This argument implicitly suggests that $X(t) = N(t)\mathbb{E}\{B\}$, which is not true. For example, if time t is a time when only two packets have arrived, then $N(t) = 2$ and $X(t) = B_1 + B_2$, which is typically much different from $2\mathbb{E}\{B\}$. This argument also does not use the law of large numbers in any way. However, the resulting answer is unfortunately correct (provided that “with probability 1” were added). Please do not confuse a correct final answer with a correct argument.

b) This argument makes no sense. The summation $\sum_{i=1}^t B_i$ is meaningless and incorrect. What does it mean to sum to t ? What if $t = 0.5$? What does a sum from $i = 1$ to 0.5 mean? But if t is an integer, the argument would *still* be incorrect. For example, let $t = 5$. But $X(5)$ has nothing to do with $\sum_{i=1}^5 B_i$, as it may not be the case that 5 packets have arrived by time 5. Further, the final answer given is clearly wrong, as it does not involve the arrival rate λ . Finally, the phrase “by the law of large numbers” is used, but clearly the law of large numbers has not been used in any way in this argument.

II. STATING THE LAW OF LARGE NUMBERS

Let $\{X_i\}_{i=1}^{\infty}$ be an infinite sequence of i.i.d. random variables. Suppose X_1 has a well defined mean (and hence all variables X_i have the same mean, by the i.i.d. assumption), and let $\mathbb{E}\{X\}$ represent this mean. Then:

$$\lim_{k \rightarrow \infty} \frac{1}{k} \sum_{i=1}^k X_i = \mathbb{E}\{X\} \quad \text{with probability 1}$$

III. PRACTICE WITH THE LAW OF LARGE NUMBERS

a) The picture is the same as given in Lecture 5, where we have a random sawtooth process (or “iid triangles”).

b) Let t_i represent the ending time of the i th triangle, which represents a renewal time because a new triangle will be chosen at that time, with a size that is independent of past history. Let $N(t)$ represent the total number of complete triangles that have ended up to and including time t (so that $N(t)$ is an integer). Let Z_i be the area of the i th triangle. Note that the total duration of the i th triangle is $H_i\alpha$.

c) The integral of $\mu(t)$ over the interval $0 \leq t \leq t_k$ is just the sum of the areas of the first k triangles:

$$\int_0^{t_k} \mu(\tau) d\tau = \sum_{i=1}^k Z_i$$

where $Z_i = \frac{\alpha}{2} H_i^2$ is the area of triangle i . Note that $\{Z_i\}$ are i.i.d. variables.

d) We have:

$$\frac{1}{t_k} \int_0^{t_k} \mu(\tau) d\tau = \frac{\sum_{i=1}^k Z_i}{\sum_{i=1}^k (H_i\alpha)} = \frac{\frac{1}{k} \sum_{i=1}^k Z_i}{\frac{1}{k} \sum_{i=1}^k (H_i\alpha)}$$

Thus (because $t_k \rightarrow \infty$ as $k \rightarrow \infty$):

$$\begin{aligned} \lim_{k \rightarrow \infty} \frac{1}{t_k} \int_0^{t_k} \mu(\tau) d\tau &= \lim_{k \rightarrow \infty} \frac{\frac{1}{k} \sum_{i=1}^k Z_i}{\frac{1}{k} \sum_{i=1}^k (H_i\alpha)} \\ &= \frac{\lim_{k \rightarrow \infty} \frac{1}{k} \sum_{i=1}^k Z_i}{\lim_{k \rightarrow \infty} \frac{1}{k} \sum_{i=1}^k (H_i\alpha)} \\ &= \frac{\mathbb{E}\{Z_1\}}{\mathbb{E}\{\alpha H_1\}} \quad \text{w.p.1 (by LLN)} \end{aligned}$$

Thus, $\bar{\mu} = \mathbb{E}\{Z_1\} / \mathbb{E}\{\alpha H_1\}$. But $\mathbb{E}\{Z_1\} = \mathbb{E}\{\alpha H_1^2 / 2\} = (\alpha/2)(p_1(25) + p_2(100))$. Also, $\mathbb{E}\{\alpha H_1\} = \alpha(p_1(5) + p_2(10))$. Therefore:

$$\bar{\mu} = \frac{(25)p_1 + (100)p_2}{2(5p_1 + 10p_2)} \quad (\text{with units of kb/sec})$$

e) The answer in part (d) yields $\bar{\mu} = 2.5$ kb/sec when $p_1 = 1$. This is intuitive because $\mu(t)$ is just a deterministic sawtooth function in this case, with average equal to half the height of the triangles. Similarly, $\bar{\mu} = 5$ kb/sec when $p_2 = 1$, as $\mu(t)$ is deterministic with triangles that are twice as large in this case.

f) If $p_1 = p_2 = 0.5$, then $\bar{\mu} = \frac{125}{30} \approx 4.1667$ from part (d). This is *larger* than 3.75 (which is the average of the two answers in part (e)), because more time is spent in triangles that are larger. Thus, larger triangles affect the time average in two ways: first, they provide larger instantaneous $\mu(t)$ values, second, they take up more of the timeline.

IV. MORE PRACTICE WITH THE LAW OF LARGE NUMBERS (WITH LESS TRAINING WHEELS)

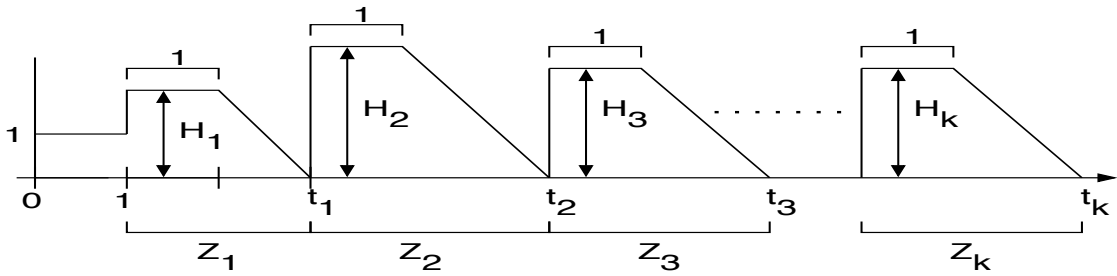


Fig. 1. A timeline of $\mu(t)$ with relevant times labeled.

The figure with appropriate labels is given in Fig.1. Let Z_i be the total duration of the trapezoid i , and let A_i be the total area of trapezoid i . Let t_i denote the ending time of trapezoid i , which also represents a renewal time (as a new i.i.d. trapezoid begins at this time).

Then:

$$\frac{1}{t_k} \int_0^{t_k} \mu(\tau) d\tau = \frac{1 + \sum_{i=1}^k A_i}{1 + \sum_{i=1}^k Z_i} = \frac{(1/k) + \frac{1}{k} \sum_{i=1}^k A_i}{(1/k) + \frac{1}{k} \sum_{i=1}^k Z_i}$$

Taking a limit as $k \rightarrow \infty$ (so that $t_k \rightarrow \infty$) thus yields:

$$\lim_{t_k \rightarrow \infty} \frac{1}{t_k} \int_0^{t_k} \mu(\tau) d\tau = \lim_{k \rightarrow \infty} \frac{(1/k) + \frac{1}{k} \sum_{i=1}^k A_i}{(1/k) + \frac{1}{k} \sum_{i=1}^k Z_i} = \frac{\mathbb{E}\{A_1\}}{\mathbb{E}\{Z_1\}} \quad \text{w.p.1 (by LLN)}$$

where we have used LLN for the numerator and denominator of the above limit, together with the fact that $\lim_{k \rightarrow \infty} 1/k = 0$. Thus, $\bar{\mu} = \mathbb{E}\{A_1\} / \mathbb{E}\{Z_1\}$. We have:

$$\begin{aligned} \mathbb{E}\{A_1\} &= \mathbb{E}\{H_1 + H_1^2 / (2\alpha)\} = \mathbb{E}\{H\} + \frac{\mathbb{E}\{H^2\}}{2\alpha} \\ \mathbb{E}\{Z_1\} &= \mathbb{E}\{1 + H_1 / \alpha\} = 1 + \frac{\mathbb{E}\{H\}}{\alpha} \end{aligned}$$

Thus:

$$\bar{\mu} = \frac{\mathbb{E}\{H\} + \frac{\mathbb{E}\{H^2\}}{2\alpha}}{1 + \frac{\mathbb{E}\{H\}}{\alpha}}$$

A note about renewal theory: The above calculation proves that:

$$\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t \mu(\tau) d\tau = \frac{\mathbb{E}\{A_1\}}{\mathbb{E}\{Z_1\}} \quad \text{with prob. 1}$$

Taking time averages using the law of large numbers in this way is one of the most important concepts of *renewal theory*. That the time average is (with prob. 1) equal to the the expected area during a renewal period divided by the expected duration of the renewal period is called the *renewal-reward theorem*. This theorem is just a direct application of the law of large numbers, and the proof is essentially the same calculation as above. *Important theorems can have very simple proofs!* More on renewal theory can be found in the renewal theory chapter of the Ross textbook.

V. THE DYNAMIC ROUTING CONJECTURE

a) The first packet that arrives to the system goes to queue 1. It exits in time 1 second. We want to design T, ϵ so that the next four packets in the batch arrive before the first packet exits queue 1 (so they all go to queue 2), and such that $1 < T$ so the next batch starts with queue 1 empty again. Choose T and ϵ such that $T > 1$ and $4\epsilon < 1$ ensures this. Then, the first packet in a batch always sees queue 1 empty and hence is placed in queue 1, and the next four packets in the batch always go to queue 2.

The rate of this arrival process is $r = 5/T$ kb/sec. The rate r_2 of data entering queue 2 is $r_2 = 4/T$ kb/sec. We want to design T such that $5/T < 4$ (so it is possible to stabilize the system with a good routing algorithm) but such that $4/T > 3$ (so that queue 2 is unstable under this particular algorithm).

Let $T = 1.26$. Then $r = 5/1.26 = 3.968$ kb/sec (so $r < 4$ kb/sec), and $r_2 = 3.175$ kb/sec. Because $r_2 > 3$ kb/sec, we have that queue 2 is unstable under this algorithm.

Finally, just choose $\epsilon = 1/8$. This ensures all four packets of a batch arrive before queue 1 empties.

b) Let $X_1(t)$ be the resulting process where we route every 4th packet to queue 1, and let $X_2(t)$ be the process of the other packets. Let $N_1(t)$ and $N_2(t)$ be the number of packets arriving to queue 1 and queue 2 (so that $X_1(t) = N_1(t)B$ and $X_2(t) = N_2(t)B$).

Let $N(t) = N_1(t) + N_2(t)$ be the total number of packet arrivals. Note that for any interval $[t_1, t_2]$ we have:

$$N[t_1, t_2] \leq 5(t_2 - t_1)/T + 5$$

and hence:

$$\begin{aligned} N_1[t_1, t_2] &\leq N[t_1, t_2]/4 + 1 \leq \frac{5}{4T}(t_2 - t_1) + 5/4 + 1 \\ N_2[t_1, t_2] &\leq N[t_1, t_2](3/4) + 3 \leq \frac{15}{4T}(t_2 - t_1) + (15/4) + 3 \end{aligned}$$

Thus (because $X_i(t) = BN_i(t)$):

$$X_1(t) \sim (r_1, \sigma_1) , \quad X_2(t) \sim (r_2, \sigma_2)$$

where $r_1 = 5B/(4T)$, $\sigma_1 = 9B/4$, $r_2 = 15B/(4T)$, $\sigma_2 = 27B/4$. Because $r_1 \leq \mu_1$ and $r_2 \leq \mu_2$, we have that for all t :

$$U_1(t) \leq \sigma_1 , \quad U_2(t) \leq \sigma_2$$

c) As we know, the JSQ policy with $K = 2$ ensures that:

$$U_1(t) + U_2(t) \leq U_{single}(t) + B_{max} \quad \text{for all } t$$

where $U_{single}(t)$ represents the unfinished work in a single server queue with the same input and with $\mu = \mu_1 + \mu_2 = 4$ kb/sec. The input process $X(t)$ is leaky bucket with parameters (r, σ) , where $r = 5/T$ kb/sec and $\sigma = 5$ kb. Thus $U_{single}(t) \leq \sigma$ for all t (note that $5/T \leq \mu = 4$).

$$U_1(t) + U_2(t) \leq \sigma + B = 6kb$$

VI. THE LEAKY BUCKET CONSTRAINT

a) Let $1.5 + z$ be the time the 3rd packet can arrive. Then we need:

$$X[1, 1.5 + z] = 21.5 \leq r(1.5 + z - 1) + \sigma = 8(0.5 + z) + 11$$

Thus, we need $z \geq (21.5 - 11 - 4)/8 = 0.8125$. But we also need:

$$X[1.5, 1.5 + z] = 11.5 \leq rz + \sigma = 8z + 11$$

Thus, we need $z \geq (11.5 - 11)/8 = 0.0625$.

Thus, the earliest time is $1.5 + 0.8125 = 2.3125$.

b) Let $1.5 + z$ be the time the 3rd packet can arrive. Then we need:

$$X[1, 1.5 + z] = 21.5 \leq r(1.5 + z - 1) + 11 = (20.5)(0.5 + z) + 11$$

and hence we need $z \geq 0.0121951$. But we also need:

$$X[1.5, 1.5 + z] = 11.5 \leq rz + \sigma = (20.5)z + 11$$

and hence we need $z \geq 0.0243902$. Thus, the earliest time is $1.5 + 0.0243902 = 1.52439$. Note in this case that the second constraint is the limiting one, different from part (a).

VII. A QUEUE WITH NON-ZERO UNFINISHED WORK

Let t_e be the time the system first empties. Thus:

$$U_0 + X[0, t_e] = \mu t_e$$

That is because the system is busy until it empties, at which point the total bits served is the total bits that arrived plus U_0 . Hence, using the fact that $X(t) \sim (r, \sigma)$:

$$\begin{aligned} \mu t_e &= U_0 + X[0, t_e] \\ &\leq U_0 + r t_e + \sigma \end{aligned}$$

Thus:

$$t_e(\mu - r) \leq U_0 + \sigma$$

and thus (because $\mu - r > 0$):

$$t_e \leq \frac{U_0 + \sigma}{(\mu - r)}$$

Defining $t^* = (U_0 + \sigma)/(\mu - r)$ yields the result.

VIII. PERIODICALLY SPLITTING A LEAKY BUCKET STREAM

a) Suppose a packet (of size B) arrives at time t_1 . Then we have $X[t_1 - \epsilon/2, t_1 + \epsilon/2] \geq B$ for any $\epsilon > 0$. But for the leaky bucket constraints to be satisfied, we need $X[t_1 - \epsilon/2, t_1 + \epsilon/2] \leq r\epsilon + \sigma$. Thus, it is necessary to have:

$$B \leq r\epsilon + \sigma$$

This must be true for all $\epsilon > 0$, and hence (taking a limit as $\epsilon \rightarrow 0$) we need $B \leq \sigma$.

b) Let $N(t)$ and $\tilde{N}(t)$ be the number of packets arriving in the original stream up to time t , and the number of packets routed to the new stream up to time t . Because every 5th packet is routed to the new stream, we have (for any interval $[t_1, t_2]$):

$$\tilde{N}[t_1, t_2] \leq N[t_1, t_2]/5 + 1$$

Also note that $X(t) = BN(t)$ and $\tilde{X}(t) = B\tilde{N}(t)$. Thus:

$$\begin{aligned} \tilde{X}[t_1, t_2] &\leq X[t_1, t_2]/5 + B \\ &\leq [r(t_2 - t_1) + \sigma]/5 + B \\ &= (r/5)(t_2 - t_1) + [\sigma/5 + B] \end{aligned}$$

and so $\tilde{X}(t) \sim (r/5, \tilde{\sigma})$, where $\tilde{\sigma} = \sigma/5 + B$.

IX. OPPORTUNISTIC SCHEDULING

a) Every slot t , we have $\mu_i(t) = 1$ with probability $(1/3)p$ (the probability that we choose queue $i \in \{1, 2, 3\}$ multiplied by the probability that channel i is ON). Thus, $\bar{\mu}_i = p/3$ kb/slot. Thus, we have rate stability under this algorithm if and only if $r \leq p/3$.

b) Note that the maximum output rate of the system is the probability that at least one channel is ON times 1 kb/sec. We thus need:

$$3r \leq Pr[\text{at least one channel ON}] = 1 - (1 - p)^3$$

and so we need $r \leq \frac{1-(1-p)^3}{3}$. We now show that this is achievable, with the following algorithm:

Algorithm: On each slot t , observe which channels are ON, and randomly and uniformly choose to serve any of the ON channels.

Let $\mu_i(t)$ be 1 if we serve channel i , and let $\mu(t) = \mu_1(t) + \mu_2(t) + \mu_3(t)$. Then $\mathbb{E}\{\mu(t)\} = Pr[\text{at least one channel is ON}]$, and $\mathbb{E}\{\mu_i(t)\} = Pr[\text{we serve channel } i]$. However:

$$1 - (1 - p)^3 = \mathbb{E}\{\mu(t)\} = \mathbb{E}\{\mu_1(t)\} + \mathbb{E}\{\mu_2(t)\} + \mathbb{E}\{\mu_3(t)\} = 3\mathbb{E}\{\mu_i(t)\}$$

where we have $\mathbb{E}\{\mu_1(t)\} = \mathbb{E}\{\mu_2(t)\} = \mathbb{E}\{\mu_3(t)\}$ by symmetry. Thus:

$$Pr[S_i(t) = ON] = \mathbb{E}\{\mu_i(t)\} = \frac{1 - (1 - p)^3}{3}$$

It follows that:

$$\bar{\mu}_i \triangleq \frac{1 - (1 - p)^3}{3}$$

and so we have stability when $r \leq \frac{1-(1-p)^3}{3}$.

c) We need $r_1 \leq Pr[\text{Channel 1 is ON}]$ and $r_1 + r_2 \leq Pr[\text{at least one channel is ON}]$, and hence we need $r_1 \leq p_1$ and $r_1 + r_2 \leq p_1 + (1 - p_1)p_2$.

Now suppose we have a vector (r_1, r_2) that satisfies these constraints. That is, suppose:

$$r_1 \leq p_1 \tag{1}$$

$$r_1 + r_2 \leq p_1 + (1 - p_1)p_2 \tag{2}$$

We must design a stabilizing algorithm.

Algorithm:

Scheduling: Every slot t , observe both channel states. Serve channel 1 whenever it is ON. Serve channel 2 else.

Routing: Route packets from stream 2 to queue 1 with probability θ_1 and to queue 2 with probability θ_2 , independently each packet.

Analysis of Algorithm:

Clearly $\bar{\mu}_1 = Pr[\text{serve 1 when it is ON}] = p_1$. Also $\bar{\mu}_2 = Pr[\text{serve 2 when it is ON}] = (1 - p_1)p_2$.

Further, the total input rate to queue 1 is $r_1 + \theta_1 r_2$, and the total input rate to queue 2 is $r_2 \theta_2$. We thus need to design θ_1, θ_2 such that:

$$r_1 + \theta_1 r_2 \leq p_1 \tag{3}$$

$$r_2 \theta_2 \leq (1 - p_1)p_2 \tag{4}$$

- If $r_1 + r_2 \leq p_1$, then choose $\theta_1 = 1$. Then clearly the constraints (3)-(4) hold.
- If $r_1 + r_2 > p_1$, choose θ_1 such that $r_1 + r_2 \theta_1 = p_1$ (this can be done with some probability θ_1 such that $0 \leq \theta_1 \leq p_1$ because $r_1 \leq p_1$). Thus, $\theta_1 = (p_1 - r_1)/r_2$. Now clearly (4) is satisfied. Let $\theta_2 = 1 - \theta_1$. Then:

$$\begin{aligned} r_2 \theta_2 &= r_2(1 - \theta_1) \\ &= r_2(1 - (p_1 - r_1)/r_2) \\ &= r_1 + r_2 - p_1 \\ &\leq p_1 + (1 - p_1)p_2 - p_1 \end{aligned} \tag{5}$$

$$= (1 - p_1)p_2 \tag{6}$$

where (5) follows from (2). Thus, (6) implies that (4) is satisfied.