EE549: Problem Set #1 Solutions

I. Drawing Basic Functions

a, b)

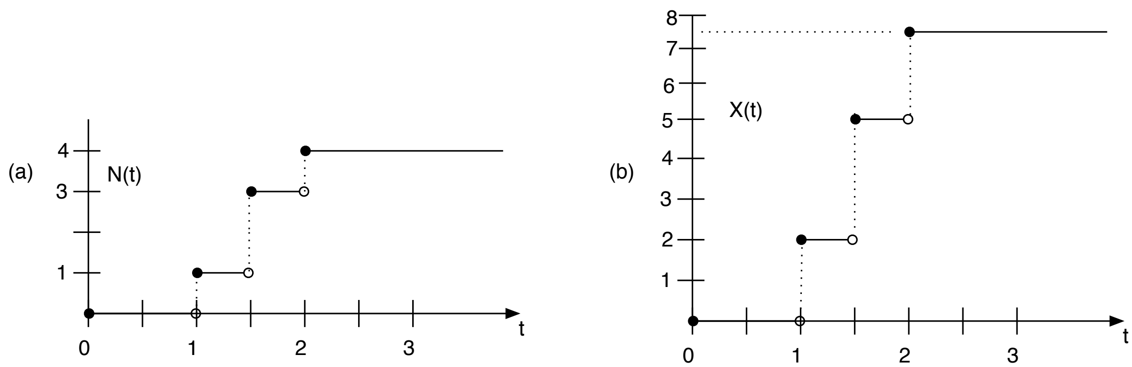

Fig. 1. The figures for problem 1(a) and 1(b).

c) Let $N(t)$ be defined according to the following arrival times: The first 42 packets arrive in bulk at time $t = 5$. Then two packets arrive periodically every second thereafter, at times $t = \{6, 7, 8, \ldots\}$. It follows that $N(5) = 42$, $N(6) = 44$, $N(7) = 46$, etc., so that $N(5 + i) = 42 + 2i$ (for $i \in \{1, 2, 3, \ldots\}$). Let $t_i = 5 + i$. Thus, $t_i \to \infty$ as $i \to \infty$, and we have:

$$\lim_{i\to\infty} \frac{N(t_i)}{t_i} = \lim_{i\to\infty} \frac{42 + 2i}{5 + i} = 2$$

Therefore, $N(t)$ has rate $\lambda = 2$ packet/sec, and $N(5) = 42$.

II. Rates and the Law of Large Numbers

a) Group consecutive inter-arrival times into pairs and define Z_i as the sum of the inter-arrival times corresponding to the ith pair. Thus: $Z_i = \tau_{2i-1} + \tau_{2i}$ for $i \in \{1, 2, \ldots, \}$. Let t_k be the time at which the kth pair ends (so that $t_k = Z_1 + Z_2 + \ldots + Z_k$). There are two arrivals for every pair, and thus:

$$\frac{N(t_k)}{t_k} = \frac{2k}{Z_1 + \ldots + Z_k} = \frac{2}{\frac{1}{k}\sum_{i=1}^{k} Z_i}$$

Note that the variables $\{Z_i\}$ are i.i.d. with mean $\mathbb{E}\{Z_i\} = \mathbb{E}\{\tau_1\} + \mathbb{E}\{\tau_2\}$. Thus, taking a limit as $k \to \infty$ and using the law of large numbers, we have:

$$\lim_{k\to\infty} \frac{N(t_k)}{t_k} = \frac{2}{\lim_{k\to\infty} \frac{1}{k}\sum_{i=1}^{k} Z_i} = \frac{2}{\mathbb{E}\{\tau_1\} + \mathbb{E}\{\tau_2\}} \text{ with prob. 1}$$

But $\mathbb{E}\{\tau_1\} = 5$, and $\mathbb{E}\{\tau_2\} = 3$. Thus, $N(t)$ has rate $\lambda = 2/(5 + 3) = 1/4$ packets/sec.

b) Partition the timeline into pairs of inter-arrival times, so that we define $W_i = Y_{2i-1} + Y_{2i}$ as the duration of the ith pair. For $i \in \{1, 2, 3, \ldots, \}$, define t_i as the time at which the ith pair ends (so that $t_i = W_1 + \ldots + W_i$). Define $t_0 = 0$. Let H_i be the integral of $\mu(t)$ over the duration of time corresponding to the ith pair. Specifically:

$$H_i = \int_{t_{i-1}}^{t_i} \mu(\tau)d\tau \text{ for } i \in \{1, 2, \ldots, \}$$

Thus

$$\frac{1}{t_k}\int_0^{t_k} \mu(\tau)d\tau = \frac{1}{\sum_{i=1}^{k} W_i}\sum_{i=1}^{k} H_i = \frac{\frac{1}{k}\sum_{i=1}^{k} H_i}{\frac{1}{k}\sum_{i=1}^{k} W_i}$$

Now note that $H_i = Z_{2i}$ for $i \in \{1, 2, \ldots\}$ (as $\mu(t) = 0$ over odd slots), where we recall that Z_i is defined as the integral of $\mu(t)$ over the ith inter-arrival time. Hence $\{H_i\}$ are i.i.d. with mean $\mathbb{E}\{H_i\} = \mathbb{E}\{Z_2\}$. Likewise, $\{W_i\}$ are i.i.d. with mean $\mathbb{E}\{W_i\} = \mathbb{E}\{Y_1\} + \mathbb{E}\{Y_2\} = 2\mathbb{E}\{Y\}$. Thus, taking a limit as $k \to \infty$ and using the law of large numbers (LLN) yields:

$$\lim_{k\to\infty} \frac{1}{t_k}\int_0^{t_k} \mu(\tau)d\tau = \frac{\lim_{k\to\infty}\frac{1}{k}\sum_{i=1}^k H_i}{\lim_{k\to\infty}\frac{1}{k}\sum_{i=1}^k W_i} = \frac{\mathbb{E}\{H_1\}}{2\mathbb{E}\{Y\}} \quad \text{with prob. 1}$$

However, $H_1 = Z_2 = \mu_2 Y_2$, and hence $\mathbb{E}\{H_1\} = \mathbb{E}\{\mu_2 Y_2\} = \mathbb{E}\{\mu_2\}\mathbb{E}\{Y\}$ (where the last equality holds because μ_2 is independent of Y_2). Further, we have $\mathbb{E}\{\mu_2\} = (10)/2 + (5)/2 = 7.5$. Thus:

$$\overline{\mu} = \frac{7.5\mathbb{E}\{Y\}}{2\mathbb{E}\{Y\}} = 3.75$$

No transmission rate units are given, but for simplicity we can assume units of kb/sec, in which case we have $\overline{\mu} = 3.75$ kb/sec.

III. More on Rates

a)

$$\lim_{t\to\infty}\frac{X_1(t) + X_2(t)}{t} = \lim_{t\to\infty}\frac{X(t)}{t} + \lim_{t\to\infty}\frac{X_2(t)}{t} = r_1 + r_2 \quad \text{with prob. 1}$$

Thus, the rate of $X_1(t) + X_2(t)$ is just the sum of the rates $(r_1 + r_2)$.

b) For any time t, we have:

$$\frac{N_1(t)}{t} = \frac{N_1(t)}{N(t)}\frac{N(t)}{t}$$

Thus:

$$\lim_{t\to\infty}\frac{N_1(t)}{t} = \lim_{t\to\infty}\frac{N_1(t)}{N(t)}\lim_{t\to\infty}\frac{N(t)}{t} = \left(\lim_{t\to\infty}\frac{N_1(t)}{N(t)}\right)\lambda \text{ with prob. 1}$$

where we have used the fact that $\lim_{t\to\infty} N(t)/t = \lambda$ with prob. 1 (as we are given that $N(t)$ has rate λ). Finally, by the law of large numbers, we have $\lim_{t\to\infty} N_1(t)/N(t) = p$ with probability 1. Therefore the rate of $N_1(t)$ is λp. Similarly, the rate of $N_2(t)$ is $\lambda(1 - p)$.

Note: How does the fact that $\lim_{t\to\infty} N_1(t)/N(t) = p$ with prob. 1 follow from the law of large numbers? This is just an average of the number of successes in the first $N(t)$ experiments, and hence converges to the probability of success p (with probability 1) as $N(t) \to \infty$. Formally, we can define 1_i as an *indicator variable*, being a random variable that is 1 if packet i is included in stream 1, and 0 otherwise. Note that $\{1_i\}_{i=1}^{\infty}$ are i.i.d. with mean $\mathbb{E}\{1_i\} = p$. Thus:

$$\lim_{t\to\infty}\frac{N_1(t)}{N(t)} = \lim_{t\to\infty}\frac{\sum_{i=1}^{N(t)} 1_i}{N(t)} = \mathbb{E}\{1_1\} = p \quad \text{with prob. 1}$$

where we have used the fact that $N(t) \to \infty$ as $t \to \infty$.

c) The following fact shall be useful for the example below:

Lemma 1: For any value $r \neq 1$, we have:

$$\sum_{n=0}^{k} r^n = \frac{1 - r^{k+1}}{1 - r}$$

Thus, $2^0 + 2^1 + \ldots + 2^{i-1} = 2^i - 1$. □

Let us now define the arrival function $N(t)$. For $i \in \{1, 2, 3, \ldots, \}$, define:

$$v_i = 2^{i-1}$$

$$t_i = v_1 + v_2 + \ldots v_i = 2^0 + 2^1 + \ldots + 2^{i-1} = 2^i - 1$$

Let $N(t)$ have arrivals as follows:

- There is 1 arrival at time $t_1 = 1$.
- There are 2 arrivals at time $t_2 = 1 + 2 = 3$.
- There are 4 arrivals at time $t_3 = 1 + 2 + 4 = 7$.
- In general, there are 2^{i-1} arrivals at time t_i.

Sequence 1: Now consider the limit of $N(t)/t$ as taken over the sequence of times $\{t_i\}$ as defined above. Note that $t_i \to \infty$ as $i \to \infty$. Further, for all $i \in \{1, 2, 3, \ldots\}$ we have:

$$\frac{N(t_i)}{t_i} = \frac{2^0 + 2^1 + 2^2 + \ldots + 2^{i-1}}{t_i} = \frac{2^i - 1}{t_i} = 1$$

Thus, $\lim_{i\to\infty} N(t_i)/t_i = 1$.

Sequence 2: Now consider the limit of $N(t)/t$ as taken over a different sequence of times. Define $a_i = t_i - 1$. Thus, a_i is the time taken one second before t_i, and we have $a_i \to \infty$ as $i \to \infty$. Note then that $N(a_i)$ does not include the 2^{i-1} packets that arrive at time t_i, but includes the $2^0 + 2^2 + \ldots + 2^{i-2}$ packets that arrived at all earlier times t_k (for $k \in \{1, \ldots, i-1\}$). Thus:

$$\begin{aligned}\frac{N(a_i)}{a_i} &= \frac{2^0 + 2^1 + \ldots + 2^{i-2}}{a_i} \\ &= \frac{2^{i-1} - 1}{t_i - 1} \\ &= \frac{2^{i-1} - 1}{2^i - 1} \\ &= \frac{1 - 1/2^{i-1}}{2 - 1/2^{i-1}}\end{aligned}$$

Therefore, $\lim_{i\to\infty} N(a_i)/a_i = 1/2 \neq 1$.

It follows that $N(t)/t$ has two different limiting values when taken over the different sub-sequences $\{t_i\}$ and $\{a_i\}$, and so the limit of $N(t)/t$ as $t \to \infty$ does not exist.

Does this problem show that our limit computations in problem 2 (which only compute the limit over a convenient set of times $\{t_i\}$), are not quite rigorous? Yes. This problem illustrates the fact that time average limits might not exist, particularly when $N(t)$ behaves strangely such as above. The above example works because the times t_i at which we sample are growing exponentially, and so there is time to "un-average" $N(t)/t$ before we get the next sample time. However, time average limits *will* exist for most systems that exhibit some "renewal" behavior, where the system repeatedly reaches some renewal state that allows application of the law of large numbers. If the limit exists, then it will be the same limit when sampled over *any* sequence of times $\{t_i\}$ (provided that $t_i \to \infty$), so that *if* the limit exists *then* we can compute it by sampling over any convenient set of times $\{t_i\}$. Thus, for simplicity, in this course we shall usually compute limits of systems with renewals by sampling over a convenient set of times $\{t_i\}$ (such as the times when the renewal events takes place, as in problem 2). A more rigorous way to compute limits in a renewal process (without looking at a particular set of times $\{t_i\}$) has been shown in Lecture 1 by "sandwiching" the limit above and below by the same thing. This simultaneously shows the limit exists and also computes what it is. But this method basically uses all the same ideas as the simpler method of sampling at convenient times $\{t_i\}$, so we shall usually use the simpler method, with the understanding that we can easily make such arguments completely rigorous by using the sandwiching method.

IV. Law of Large Numbers and Departures

a) Note that cars are taken in pairs from the infinite buffer of cars, and put into the windows 1 and 2 simultaneously. Let S_i be the time required for the ith pair of cars to depart (where $S_i = \max[X_i, Z_i]$, with X_i being the random service time at window 1 and Z_i being the independent and random service time at window 2. Thus, $\{S_i\}$ are i.i.d. with mean $\mathbb{E}\{S_1\}$. Let $K(t)$ be the number of pairs that have been completely served by time t. Then $K(t)$ is a renewal process with inter-arrival times S_i. It follows that $\lim_{t\to\infty} K(t)/t = 1/\mathbb{E}\{S_1\}$ with prob. 1. Furthermore, note that:

$$2K(t) \leq D(t) \leq 2K(t) + 1$$

Therefore:

$$\lim_{t\to\infty}\frac{D(t)}{t} = 2/\mathbb{E}\{S_1\} \quad \text{with prob. 1}$$

Because X_1 and Z_1 are i.i.d. and uniform over the interval $[0,1]$, we have $\mathbb{E}\{S_1\} = \mathbb{E}\{\max[X_1, Z_1]\} = 2/3$. Thus: $D(t)$ has departure rate 2/(2/3) = 3 cars/unit time.

Note: Why is $\mathbb{E}\{\max[X_1, Z_1]\} = 2/3$? We have:

$$\begin{aligned}
\mathbb{E}\{\max[X_1, Z_1]\} &= \int_0^1 \mathbb{E}\{\max[X_1, z] \mid Z_1 = z\}\, p_{Z_1}(z)dz \\
&= \int_0^1 \mathbb{E}\{\max[X_1, z]\}\, dz && (1)\\
&= \int_0^1 \left[z(z) + \frac{z+1}{2}(1-z)\right] dz && (2)\\
&= \left[\frac{z^3}{3} + \frac{z}{2} - \frac{1}{6}z^3\right]\Big|_0^1 && (3)\\
&= 2/3
\end{aligned}$$

where (1) follows because $\mathbb{E}\{\max[X_1, z] \mid Z_1 = z\} = \mathbb{E}\{\max[X_1, z]\}$ (because X_1 and Z_1 are independent), and (2) follows because $\mathbb{E}\{\max[X_1, z]\} = zPr[X_1 \leq z] + \frac{z+1}{2}Pr[X_1 > z]$.

b) Case 1 (The first strategy): The departure rate is $2/\mathbb{E}\{\max[X_1, Z_1]\}$, where X_1, Z_1 are independent, and X_1 is uniform over $[0,2]$ and Z_1 is uniform over $[0,3]$. It suffices to just compute $\mathbb{E}\{\max[X_1, Z_1]\}$. We have:

$$\begin{aligned}
\mathbb{E}\{\max[X_1, Z_1]\} &= \mathbb{E}\{\max[X_1, Z_1] \mid Z_1 \leq 2\}\, Pr[Z_1 \leq 2] + \mathbb{E}\{\max[X_1, Z_1] \mid Z_1 > 2\}\, Pr[Z_1 > 2] \\
&= [4/3](2/3) + \mathbb{E}\{Z_1 \mid Z_1 > 2\}\,(1/3) && (4)\\
&= \frac{8}{9} + \frac{2.5}{3} && (5)\\
&= 31/18
\end{aligned}$$

where (4) follows because $Pr[Z_1 \leq 2] = 2/3$, $Pr[Z_1 > 2] = 1/3$, and Z_1 is uniformly distributed over $[0,2]$ when it is conditioned by the even $Z_1 \leq 2$. Equation (5) follows because Z_1 is uniformly distributed over $[2,3]$ when conditioned by $Z_1 > 2$.

Thus, the departure rate of cars in this case is $2/(31/18) = 36/31 \approx 1.16129$.

Case 2 (The second strategy): In the second strategy we only use window 2, and so inter-arrival times are i.i.d. and uniformly distributed over $[0,2]$. Therefore, the departure rate under this strategy is $1/\mathbb{E}\{X_1\} = 1$ packets/unit time. This is not as good as the first strategy.

V. INPUT/OUTPUT EXAMPLES

a)

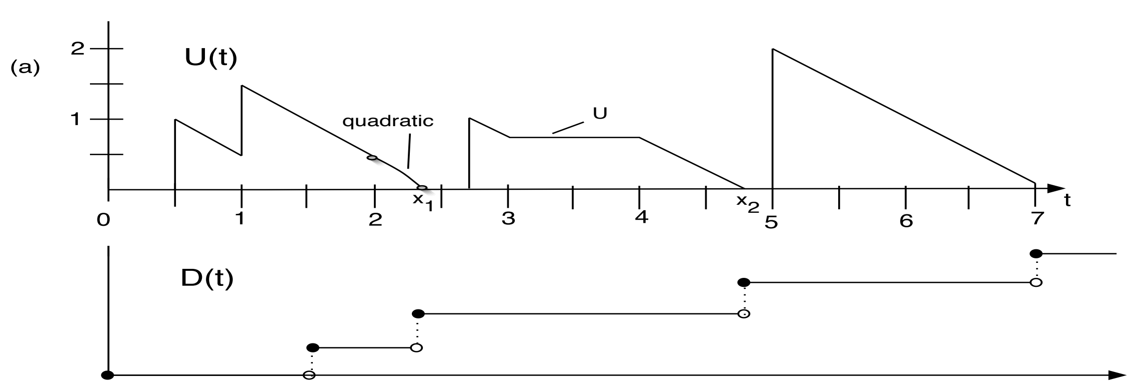

Fig. 2. The figures for problem part (a).

The figures are drawn in Fig. 2. The first busy period ends at time x_1, where $x_1 = 2 + x$, and x satisfies:

$$\int_0^x (1 + 2t)dt = 1/2$$

Thus:

$$x + x^2 = 1/2$$

Thus $x = \frac{-2+\sqrt{12}}{4} \approx 0.366$, and $x_1 \approx 2.366$.

The amount of work in the system at time $t = 3$ is U, where:

$$U = 1 - \int_0^{.3} (1.2 - 4t)dt = 1 - [(1.2)(.3) - 2(.3)^2] = 1 - 0.18 = 0.82$$

The third packet thus leaves at time $x_2 = 4.82$. The last packet leaves at time 7.

Thus, the departure times are: $\{1.5, x_1 = 2.366, x_2 = 4.82, 7\}$.

b) The departures from queue 1 are at times $\{1.5, 2.5, 5\}$, which form the jumping times of the process $D_1(t)$. The total input process to queue 2 has arrivals at times $\{1.5, 2.2, 2.5, 4.9, 5\}$. Note that the packet arrivals to this queue are of type: $\{1, 2, 1, 2, 1\}$. The departures from the entire system are at times $\{2.5, 3.5, 4.5, 5.9, 6.9\}$, which for the jump times of the $D_2(t)$ process.

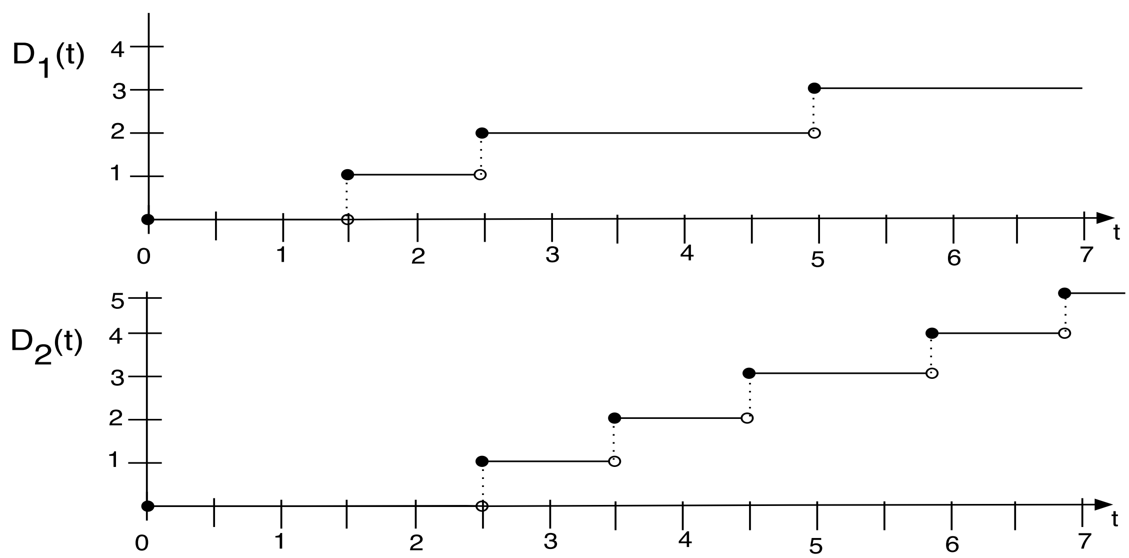

Fig. 3. The departure functions $D_1(t)$ and $D_2(t)$.

c) The type 1 packets enter queue 2 at times $\{1.5, 2, 5\}$. The total arrivals to queue 2 are at times: $\{1.5, 2, 2.2, 4.9, 5\}$, with type ordering: $\{1, 1, 2, 2, 1\}$. Note that the arrival ordering to queue 2 is different, and hence (by FIFO) the departure ordering is different from part (b). The departures from the system are at times $\{2.5, 3.5, 4.5, 5.9, 6.9\}$. These are the same departure times as part (b), and *hence the $D_2(t)$ function is identical to that in part (b) (see Fig. 3).*

Explanation: The first queue has the same service time as the second queue. Hence, by replacing the first queue with a pure time delay, we have the first queue placing packets to the second queue faster than before. But the first packet of any busy period of the first queue is placed to the second queue exactly at the same time as in the system with the pure delay. Further, any remaining packets of the busy period of the first queue are placed to the second queue on or before the time required for the second queue to have completely processed the previous packet. Thus the busy periods in the second queue are the same with or without the replacement of the first queue with a time delay. It follows that the departure process from the second queue is the same.

VI. TRUE/FALSE

- a) False: Let $U(t_1) = 0$ and suppose that no arrivals occur over the interval $[t_1, t_2]$. Let $t_2 = t_1 + T$ (where $T > 0$). Suppose $\mu(t) = 1$ for all t. Then:

$$\begin{aligned} U(t_2) &= 0 \\ U(t_1) + X(t_1, t_2] - \int_{t_1}^{t_2} \mu(\tau)d\tau &= 0 + 0 - (t_2 - t_1) = -T \neq 0 \end{aligned}$$

- b) False: For simplicity, let $\mu = 1$ kb/sec. Suppose that $X_1(t)$ consists of exactly one packet arrival of size $B = 1$ kb at time $t_1 = 70$. Suppose that $X_2(t)$ consists of exactly one packet arrival of size $B = 1$ kb at time $t_2 = 1$. Then clearly we have:

$$X_1(t) = \begin{cases} 0 & \text{if } t < 70 \\ 1 & \text{if } t \geq 70 \end{cases} \quad , \quad X_2(t) = \begin{cases} 0 & \text{if } t < 1 \\ 1 & \text{if } t \geq 1 \end{cases}$$

  It follows that $X_1(t) \leq X_2(t)$ for all t. But clearly $U_1(70) = 1$, $U_2(70) = 0$, and so $U_1(70) > U_2(70)$. We have $U_2(70) = 0$ because there is only one packet that arrives at queue 2, and this departs at time 2.
- c) True: This because $X(t_2) - X(t_1) = X(t_1, t_2]$. This does not include any bits that might arrive exactly at time t_1. But $(X(t_1^+) - X(t_1^-))$ is exactly the amount of bits that arrive exactly at time t_1.
- d) True: $X(a_n+5) - X(a_n-5) = X(a_n-5, a_n+5]$. This is the amount of bits that arrive during $(a_n-5, a_n+5]$, which includes the bits B_n that arrive at time a_n.

VII. INEQUALITY COMPARISON

a) If $U_1(t) = 0$ then we clearly have $U_1(t) = 0 \leq U_2(t)$.
b) Suppose that $U_1(t) > 0$. Then there is a time $t_b \leq t$ such that: $U_1(t) = X[t_b, t] - \int_{t_b}^t \mu(\tau)d\tau$. Therefore:

$$\begin{aligned} U_1(t) &= X[t_b, t] - \int_{t_b}^t \mu(\tau)d\tau \\ &\leq X[t_b, t] + Z[t_b, t] - \int_{t_b}^t \mu(\tau)d\tau \\ &\leq X[t_b, t] + Z[t_b, t] - Y_2[t_b, t] \quad (6) \\ &\leq U_2(t_b^-) + (X[t_b, t] + Z[t_b, t]) - Y_2[t_b, t] = U_2(t) \quad (7) \end{aligned}$$

where (6) follows because $Y_2[t_b, t] \leq \int_{t_b}^t \mu(\tau)d\tau$, and (7) follows by the I-O equation for queue 2.