EE549: Problem Set #1
Due Wednesday, Jan. 23

I. DRAWING BASIC FUNCTIONS

a) Draw the curve for the packet arrival process $N(t)$ that consists of four packets only: The first packet arrives at time $t = 1$. The next two packets arrive exactly at the same time, at $t = 1.5$. The fourth packet arrives at time $t = 2$ (the inter-arrival times are thus $\tau_1 = 1$, $\tau_2 = 0.5$, $\tau_3 = 0$, $\tau_4 = 0.5$).

b) Suppose that the packet sizes for the four packets given in part (a) are $B_1 = 2$, $B_2 = 1$, $B_3 = 2$, $B_4 = 2.5$ (in units of kilobits). Plot $X(t)$. (Recall that $X(t) = \sum_{i=1}^{N(t)} B_i$).

c) Give an example of a deterministic packet arrival process $N(t)$ such that $N(t)$ has rate $\lambda = 2$ packets/sec (i.e., $\lim_{t\to\infty} N(t)/t = 2$), and such that $N(5) = 42$. That is, you should specify when packets arrive over time, for $t \geq 0$. You can specify either the actual arrival epochs $\{t_i\}_{i=1}^{\infty}$, or the inter-arrival times $\{\tau_i\}_{i=1}^{\infty}$.

(for parts (a), (b), (c), you should consider only $t \geq 0$ and assume $N(0) = 0$, $X(0) = 0$).

II. RATES AND THE LAW OF LARGE NUMBERS

a) Let $N(t)$ be a packet arrival process with inter-arrival times that are independent but not identical. Specifically, every odd inter-arrival time τ_i (for $i \in \{1, 3, 5, 7, \ldots\}$) is a uniform random variable over the interval $[4, 6]$. Every even inter-arrival time τ_i (for $i \in \{2, 4, 6, \ldots\}$) is exponential with rate γ, where $\gamma = 1/3$ (so that the pdf is $p_\tau(v) = \gamma e^{-\gamma v}$ for $v \geq 0$). Compute the rate of $N(t)$ (in units of packets/sec). Hint: Use the law of large numbers on a suitable sequence of random variables.

b) Let $\mu(t)$ be a time varying server rate process. The server rate is constant over successive intervals of size $Y_1, Y_2, Y_3, \ldots$, where Y_i are i.i.d. with mean $\mathbb{E}\{Y\}$. On odd intervals (corresponding to $Y_1, Y_3, Y_5, \ldots$), we have $\mu(t) = 0$. On even intervals (corresponding to $Y_2, Y_4, Y_6, \ldots$), the server rate is constant $\mu(t) = \mu_i$, where μ_i is i.i.d. over each even interval i and is equal to 10 with probability $1/2$ and 5 with probability $1/2$. Compute the time average rate of $\mu(t)$, i.e., show that $\lim_{t\to\infty} \frac{1}{t}\int_0^t \mu(\tau)d\tau$ converges to a particular value $\overline{\mu}$ with probability 1. (Hint: Let $Z_2, Z_4, Z_6, \ldots$ be the value of the integral of $\mu(t)$ over the intervals corresponding to $Y_2, Y_4, Y_6, \ldots$, and apply the law of large numbers).

III. MORE ON RATES

a) Suppose that $X_1(t)$ and $X_2(t)$ are bit arrival processes with rates r_1 and r_2, respectively (in units of bits/sec). Compute the rate of the sum process $X_1(t) + X_2(t)$.

b) Let $N(t)$ be a packet arrival process with rate λ. Suppose that we split this process into two streams, 1 and 2, with corresponding arrival processes $N_1(t)$ and $N_2(t)$ (so that $N_1(t) + N_2(t) = N(t)$). Stream 1 is formed by i.i.d. *Bernoulli splitting*: Each arriving packet from $N(t)$ is independently chosen to include in stream 1 with probability p. Stream 2 is formed by all remaining packets not placed on stream 1. Prove that $N_1(t)$ and $N_2(t)$ have rates $p\lambda$ and $(1-p)\lambda$, respectively.

c) Give an example of a deterministic packet arrival process $N(t)$ that does not have a well defined rate, i.e., $\lim_{t\to\infty} \frac{N(t)}{t}$ does not exist. The best way to answer this question is to define the process $N(t)$, and describe two sequences of times $\{t_i\}$ and $\{a_i\}$ such that $t_i \to \infty$ and $a_i \to \infty$ as $i \to \infty$, but such that $\lim_{i\to\infty} \frac{N(t_i)}{t_i} \neq \lim_{i\to\infty} \frac{N(a_i)}{a_i}$. (For example: the function $f(t) = cos(t)$ does not have a well defined limit, as it has two different limits when taken over the sequences $t_i = i2\pi$ and $a_i = i2\pi + \pi$.) Note that a limit of ∞ is still a limit that exists, so the answer is not just a limit that goes to infinity.

IV. Law of Large Numbers and Departures

At a certain drive-through restaurant, there is only space for one line of cars (see Fig. 1). There are 2 server windows, each one able to take the order, payment, and give the food to a given car. The servers are aligned so that if a new car sees both windows available, it will go to the window furthest down the line (window 2). If this window is busy, it will instead go to window one, and else it will wait in a "buffer" until it can be served. Due to the narrow road, a car being served at window 1 cannot depart until the car (potentially) in service at window 2 is gone. Service times are i.i.d. and uniformly distributed between $[0, 1]$. Neglect the extra time required for a car to drive in and out of the service area. Suppose there is an infinite backlog of cars available, so that a new car enters a window whenever possible. Let $D(t)$ represent the total number of car departures from the system up to time t.

a) What is the departure rate of cars (i.e., what is $\lim_{t\to\infty} D(t)/t$?)

b) Suppose that the service times at window 2 are i.i.d. and uniformly distributed over the interval $[0, 2]$, but the service times at window 1 are i.i.d. and uniformly distributed over the interval $[0, 3]$. Once a car starts its service at a particular window, we assume that it stays at that window until the end of its service. Consider the following two strategies:

- Use both windows as before: Cars go to window 2 if it is open, and go to window 1 if it is open and window 2 is not open.
- Ignore window 1: Cars wait only for service at window 2. If window 1 is available, they just ingore it. Thus, window 2 continually serves cars.

Which strategy is better in terms of departure rate?

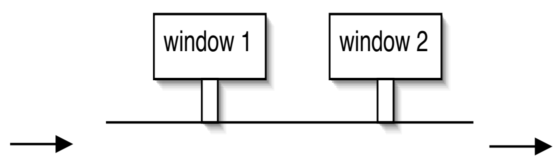

Fig. 1. The drive-through restaurant with two windows.

V. Input/Output Examples

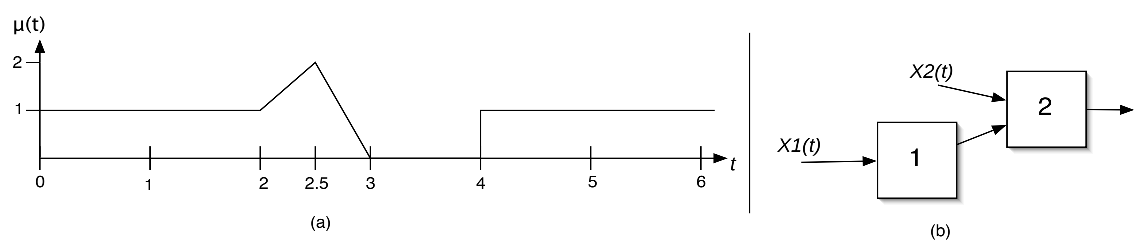

Fig. 2. (a) An example $\mu(t)$ function. (b) A tandem of 2 queues.

a) Three packets of size 1 unit arrive to a queue at times $t = 0.5, 1, 2.7$, and a packet of size 2 units arrives at time $t = 5$. The queue is initially empty at time 0, and has a $\mu(t)$ function as shown in Fig. 2a. Plot the resulting unfinished work function $U(t)$. Assuming FIFO service and that packets do not depart the system until their last bit departs, plot the resulting packet departure function $D(t)$.

b) Consider a tandem of two queues as shown in Fig. 2b. Both queues have constant server rates $\mu = 1$ kilobyte/second. Packets arrive from stream $X_1(t)$ at times $t = 0.5, 1, 4$ seconds. Packets arrive from stream $X_2(t)$ at times $t = 2.2, 4.9$ seconds. All packets have fixed lengths of 1 kilobyte. Plot the departure functions $D_1(t)$ and $D_2(t)$.

c) Remove the first queue and assume the $X_1(t)$ stream enters into the final queue after a delay of 1 second (so that the total input to the final queue is the sum process $X_1(t-1) + X_2(t)$). Plot the departure function and compare to part b. Explain. Is packet ordering preserved?

VI. TRUE/FALSE

For all problems, indicate whether the result is TRUE or FALSE. If TRUE, explain why. If FALSE, provide a counter-example.

- a) For a single server, work conserving queue with server process $\mu(t)$, we have for any two times t_1, t_2 such that $t_1 < t_2$

$$U(t_2) = U(t_1) + X(t_1, t_2] - \int_{t_1}^{t_2} \mu(\tau) d\tau$$

- b) Suppose $X_1(t)$ and $X_2(t)$ are two arrival processes such that $X_1(t) \leq X_2(t)$ for all t. $X_1(t)$ enters a work conserving queue that is initially empty and has a constant server rate μ. $X_2(t)$ also enters a work conserving queue that is initially empty and has a constant server rate μ. Then $U_1(t) \leq U_2(t)$ for all t.
- c) For any times $t_1 < t_2$, we have:

$$X[t_1, t_2] = X(t_2) - X(t_1) + (X(t_1^+) - X(t_1^-))$$

- d) Let $X(t)$ be an arrival process, and let $\{a_1, a_2, \ldots, a_n, \ldots\}$ represent the sequence of packet arrival times and $\{B_1, B_2, \ldots, B_n, \ldots\}$ represent the sequence of packet lengths. Then for every n, we have:

$$B_n \leq X(a_n + 5) - X(a_n - 5)$$

VII. INEQUALITY COMPARISON

Consider a queue with server process $\mu(t)$ and input process $X(t)$, and let $U_1(t)$ represent the resulting unfinished work function. Now let $U_2(t)$ represent the unfinished work in another queue with the same server rate $\mu(t)$ but with an input stream $X(t) + Z(t)$, where $Z(t)$ is any other arrival process. Both queues are initially empty, and both are work conserving. We want to show that $U_1(t) \leq U_2(t)$ for all $t \geq 0$.

a) Fix a time $t \geq 0$. Show that the result holds in the special case when $U_1(t) = 0$.

b) Now show that the result also holds in the opposite case when $U_1(t) > 0$. (Recall that in such a case, queue 1 must be in a busy period).