Frequency Analysis and Norms of Distributed Spatially Periodic Systems

Makan Fardad, Mihailo R. Jovanović, *Member, IEEE*, and Bassam Bamieh, *Fellow, IEEE*

***Abstract*—We investigate several fundamental aspects of the theory of linear distributed systems with spatially periodic coefficients. We develop a spatial-frequency domain representation analogous to the lifted or frequency response operator representation for linear time periodic systems. Using this representation, we introduce the notion of the $\mathcal{H}^2$ norm for this class of systems and provide algorithms for its computation. A stochastic interpretation of the $\mathcal{H}^2$ norm is given in terms of spatially cyclostationary random fields and spectral-correlation density operators. When the periodic coefficients are viewed as feedback modifications of spatially invariant systems, we show how they can stabilize or destabilize the original systems in a manner analogous to vibrational control or parametric resonance in time periodic systems. Two examples from physics are provided to illustrate the main results.**

***Index Terms*—Cyclostationary random fields, frequency domain lifting, frequency response operators, $\mathcal{H}^2$ norm, partial differential equation (PDE) with periodic coefficients, spatially periodic systems.**

I. Introduction

SPATIALLY distributed systems represent a special class of distributed parameter systems in which states, inputs, and outputs are spatially distributed fields. The general theory of distributed parameter systems is by now well developed and mature [1]–[4]. A recent trend has been the exploitation of special system structures in order to derive less conservative results than is possible for very general classes of infinite-dimensional systems. Examples of this are many, with the most closely related to our work being the class of spatially invariant systems [5]. Although the study of special classes of systems restricts the applicability of the given results, one should interpret these results in a wider context. An appropriate analogy might be the theory of linear time invariant (LTI) systems, which is widely used as a starting point in the analysis and control of systems that are in effect time varying and nonlinear.

This paper considers a class of distributed spatially periodic systems. These are spatially distributed linear dynamical systems over infinite spatial domains in which the underlying partial differential equations (PDEs) contain spatially periodic coefficients with commensurate periods. This class of systems has rich behavior in terms of response characteristics, stability properties, and signal amplification as measured by system norms. A particular objective of this study is to observe phenomena similar to the change of dynamical properties of LTI systems described by ordinary differential equations (ODEs) when temporally periodic coefficients are introduced. For example, certain unstable LTI systems can be stabilized by being placed in feedback with temporally periodic gains of properly designed amplitudes and frequencies. This can be roughly considered as an example of *vibrational control* [6]. On the other hand, certain stable or neutrally stable LTI systems can be destabilized by periodic feedback gains. This phenomenon is sometimes referred to as *parametric resonance* in the dynamical systems literature [7]. In all of these examples, the periodic terms in the ODEs can be considered as feedback modifications of an LTI system. The stabilization/destabilization process depends in subtle ways on resonances between the natural modes of the LTI subsystem and the frequency and amplitude of the periodic coefficients.

In this paper, we provide several examples in which this stabilization/destabilization process occurs in PDEs by the introduction of *spatially periodic* coefficients. In a close analogy with the ODE examples listed above, we consider spatially periodic systems as spatially invariant systems modified by feedback with spatially periodic gains. The periodicity of the coefficients can occur in a variety of ways. For example, in boundary layer and channel flow problems with corrugated walls (referred to as "riblets" in the literature), the linearized Navier-Stokes equations in this geometry have periodic coefficients. This periodicity appears to influence flow instabilities and disturbance amplification, and may ultimately explain drag reduction or enhancement in such geometries. Photonic crystals, meta-materials, and frequency selective surfaces are other examples of man-made spatially periodic structures designed to change the properties of distributed systems in a favorable way. PDEs with spatially periodic coefficients also arise in problems of pattern formation in chemical and biological systems, as well as optical and fluid systems [8]. In these systems one deals with nonlinear PDEs whose linearization around the trivial solution has unstable modes for certain values of the spatial frequency. The nonlinearity then saturates the growth of the unstable modes, leading to a bounded spatially periodic solution responsible for pattern formation. The stability of the formation can then be verified by linearization of the equation around the spatially periodic solution, which results in a PDE with spatially periodic coefficients [9]. The above applications represent significant research efforts in themselves and we do not directly address them in this paper. Instead, we illustrate the effects of periodic coefficients on stability properties and system norms of certain PDEs that are of particular interest in these applications, namely the Ginzburg–Landau and the Swift–Hohenberg equations [8].

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M. Fardad is with the Department of Electrical Engineering and Computer Science, Syracuse University, Syracuse, NY 13244 USA (e-mail: makan@syr.edu).

M. R. Jovanović is with the Department of Electrical and Computer Engineering, University of Minnesota, Minneapolis, MN 55455 USA (e-mail: mihailo@umn.edu).

B. Bamieh is with the Department of Mechanical Engineering, University of California, Santa Barbara, CA 93106 USA (e-mail: bamieh@engineering.ucsb.edu).

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The objective of this paper is to lay the foundations for the study of linear spatially periodic systems and provide more detailed tests for analysis of system theoretic properties than can be achieved by considering them as general distributed parameter systems. The main technique used is a spatial-frequency representation akin to that used for temporally periodic systems, and alternatively referred to as the *harmonic transfer function* [10], the *frequency response (FR) operator* [11], [12], or *frequency domain lifting* [13], [14]. By using a similar frequency representation in the spatial coordinate, we reduce the study of linear spatially periodic systems to that of families of operator-valued LTI systems, where the operators involved are bi-infinite matrices with a special structure. This is in contrast to spatially invariant systems, where the proper frequency representation produces families of matrix-valued LTI systems. This reflects the additional richness of the class of spatially periodic systems as well as the fact that such systems mix frequency components in a certain way. We pay special attention to the stochastic interpretation of this frequency representation by considering spatially distributed systems driven by random fields. The resulting random fields are not necessarily spatially stationary but rather spatially *cyclostationary* (a direct analog of the notion of temporally cyclostationary processes [15]). This reflects the fact that frequency components of a cyclostationary field are correlated in ways determined by the underlying system's periodicity. We show how the frequency domain representation of systems clarifies frequency mixing in the deterministic case and frequency correlations in the stochastic case.

The paper is organized as follows: We begin with a motivating example of a spatially periodic system in Section II. We review relevant definitions in Section III and present the frequency representation of spatially periodic operators in Section IV. We introduce the class of spatially periodic systems and their frequency domain representation in Section V. We define the $\mathcal{H}^2$ norm of spatially periodic systems in Section VI, derive formulas for computing it in the spatial-frequency domain, and discuss issues of truncation and finite-dimensional approximations. Illustrative examples are provided in Section VII where it is shown how stability properties and input-output norms of spatially invariant systems can be changed by the introduction of spatially periodic coefficients. To improve readability, proofs are relegated to the Appendix.

Terminology

Throughout the paper, we use the terms *spatial operators* and *spatial systems*. By the former, we mean a *purely spatial* system with no temporal dynamics (i.e., a memoryless operator that acts on a spatial function and yields a spatial function, similar to a static gain in finite-dimensional systems), whereas the latter refers to a *spatio-temporal* system (i.e., a system in which, at each time t, the state is a function on a spatial domain). Also, spatial and spatio-temporal stochastic processes will be called *random fields*, to differentiate them from purely temporal stochastic processes. The $\mathcal{H}^2_{\rm sp}$ and $\mathcal{H}^\infty_{\rm sp}$ norms are extensions of the well-known $\mathcal{H}^2$ and $\mathcal{H}^\infty$ norms of stable linear systems to purely spatial operators.

Notation

The spatial variable is denoted by $x \in \mathbb{R}$. We use $k_x \in \mathbb{R}$ to characterize the spatial-frequency variable, also known as the *wave-number*. $\mathbb{C}^-$ denotes all complex numbers with real part strictly less than zero, $\mathbb{C}^+$ denotes all complex numbers with real part greater than or equal to zero, and $j := \sqrt{-1}$. $\overline{\mathfrak{S}}$ is the closure of the set $\mathfrak{S} \subset \mathbb{C}$. For vectors (and matrices) $\|\cdot\|$ denotes the standard Euclidean norm (induced norm), and may also be used to denote the induced norm on operator spaces when there is no chance of confusion. "*" denotes the complex-conjugate transpose of a matrix and the adjoint of a linear operator. $\Sigma(T)$ is the spectrum of the operator T, and $\Sigma_{\rm p}(T)$ is its point spectrum. $L^2(\mathbb{R})$ is the space of square integrable functions on the real line and ℓ^2 is the space of square summable sequences on $\mathbb{Z}$. $\mathsf{B}(\ell^2)$ denotes the bounded operators on ℓ^2, $\mathsf{B}_0(\ell^2)$ the compact operators on ℓ^2, and $\mathsf{B}_2(\ell^2)$ the Hilbert–Schmidt operators on ℓ^2; $\mathsf{B}_2(\ell^2) \subset \mathsf{B}_0(\ell^2) \subset \mathsf{B}(\ell^2)$ [16]. We use the same notation G for a system/operator and its kernel representation. The spatio-temporal function $u(t,x)$ (operator G) is denoted by $\hat{u}(t,k_x)$ (respectively $\hat{G}$) after the application of a Fourier transform on the spatial variable x. The expected value of the spatio-temporal random field $u(t,x)$ is denoted by $\mathsf{E}\{u(t,x)\}$.

II. Motivating Example

Our aim in this section is to motivate the development in the rest of the paper by introducing an example of a spatially periodic linear system and considering the problem of its stability.

The Ginzburg–Landau (GL) equation on the real line is studied in [17]. This equation results from nonlinear stability theory and appears in the analysis of many problems in fluid mechanics [8], including the Bénard problem, the Taylor problem, Tollmien–Schlichting waves, and gravity waves. It is worth mentioning that the GL equation describes the evolution of a slowly varying complex amplitude of a neutral plane wave. For more details see [8], [17] and references therein.

The linearization of this equation around its limit-cycle solution $\phi_0(t,x) := f(x)\exp(j\omega_0 t)$, $f(x) = f(x + \mathrm{X})$, results in a PDE with periodic coefficients of the form

$$\begin{aligned} \partial_t \phi &= z_1\phi + z_2\partial_x^2\phi + z_3(2|f|^2\phi + f^2\phi^*) \\ y &= \phi. \end{aligned} \tag{1}$$

The variable x denotes the spatial coordinate and belongs to $\mathbb{R}$. The spatio-temporal function ϕ represents the state and belongs to the space $L^2(\mathbb{R})$ at any given time t, the function ϕ^* is the complex-conjugate of ϕ, and

$$z_1 := \rho - j\omega_0, \quad z_2 := c_0 + j, \quad z_3 := j - \rho \tag{2}$$

with c_0, ω_0, ρ being real-valued parameters. Furthermore $\rho := c_0/c_1$ and $0 \le c_0^2 \le c_1$.

If we introduce the following notation:

$$\psi := \begin{bmatrix} \phi \\ \phi^* \end{bmatrix} \tag{3}$$

we can rewrite (1) as

$$\partial_t \psi = A\psi. \tag{4}$$

The operator A in this equation is given by

$$A = \begin{bmatrix} z_1 + z_2\partial_x^2 + 2z_3|f|^2 & z_3 f^2 \\ (z_3 f^2)^* & z_1^* + z_2^*\partial_x^2 + 2z_3^*|f|^2 \end{bmatrix} \tag{5}$$

where f is a spatially periodic function, $f(x) = f(x + \mathrm{X})$.

Suppose we are interested in the stability of the PDE described by (4)–(5). If $f \equiv 0$, then A contains only spatial derivatives and is therefore a *spatially invariant* operator (i.e., it commutes with all spatial shifts). One can then use a Fourier transformation in the variable x to convert operator A to a multiplication operator; effectively the Fourier transform "diagonalizes" the operator A. Let $\hat{A}(k_x)$, $k_x \in \mathbb{R}$, denote the Fourier symbol of A. Then system (4) is stable if and only if for every $k_x \in \mathbb{R}$ the eigenvalues of $\hat{A}(k_x)$ remain in the left-half of the complex plane and bounded away from the imaginary axis (under some additional technical assumptions [5]). This "decoupling" in the frequency variable k_x is what we refer to as diagonalization in the Fourier domain. This property can be utilized to significantly simplify not only the analysis but also the design of controllers for spatially invariant systems [5].

In the case where f is a non-constant spatially periodic function, the operator A is no longer spatially invariant and thus the Fourier transform no longer diagonalizes A. Nevertheless, the Fourier transform of the *spatially periodic* operator A continues to have a special structure. In the rest of the paper we describe this structure and explain how it can be exploited to convert the problem to a form amenable to the application of system theoretic concepts.

III. Preliminaries

In this section we briefly review some of the definitions used throughout the paper.

A. Spatial Operators and Spatial Systems

We consider spatial operators that can be described by a *kernel function*; we assume that if u and y are two spatial functions related by a linear operator G, then the relation between them is described by the equation

$$y(x) = \int_{\mathbb{R}} G(x,\chi)u(\chi)d\chi \tag{6}$$

where with an abuse of notation we use G to represent both the operator and its kernel function. A linear *spatially periodic operator* with period X is one whose kernel has the property

$$G(x+\mathrm{X}m, \chi+\mathrm{X}m) = G(x,\chi)$$

for all $x,\ \chi \in \mathbb{R}$ and $m \in \mathbb{Z}$.

We consider spatially distributed systems for which the spatio-temporal input u and spatio-temporal output y are related by the equation

$$y(t,x) = \int_0^t \int_{\mathbb{R}} G(t,\tau;x,\chi)u(\tau,\chi)d\chi d\tau. \tag{7}$$

A spatially periodic LTI system with spatial period X, is one whose kernel satisfies

$$G(t+s,\tau+s;x+\mathrm{X}m,\chi+\mathrm{X}m) = G(t,\tau;x,\chi)$$

for all $t,\ \tau,\ s \in [0,\infty)$, $t \geq \tau$, $x,\ \chi \in \mathbb{R}$ and $m \in \mathbb{Z}$. From time-invariance it follows that

$$G(t,\tau;x,\chi) = H(t-\tau;x,\chi)$$

for all $x,\ \chi \in \mathbb{R}$, where H is the temporal (operator-valued) impulse response of the system, i.e., (7) can also be expressed as the temporal convolution $y(t,x) = \int_0^t \int_{\mathbb{R}} H(t-\tau;x,\chi)u(\tau,\chi)\ d\chi\ d\tau$.

B. Stochastic Processes and Random Fields

A spatial random field v is called *wide-sense cyclostationary* if its autocorrelation $R^v(x,\chi) := \mathsf{E}\{v(x)v^*(\chi)\}$ satisfies

$$R^v(x+\mathrm{X}m,\chi+\mathrm{X}m) = R^v(x,\chi) \tag{8}$$

for all $x,\chi \in \mathbb{R}$, $m \in \mathbb{Z}$, and some $\mathrm{X} \in \mathbb{R}$.

Let u denote a spatio-temporal random field with autocorrelation $R^u(t,\tau;x,\chi) := \mathsf{E}\{u(t,x)u^*(\tau,\chi)\}$. If $u(t,x)$ is wide-sense stationary in both the temporal and spatial directions, then

$$R^u(t,\tau;x,\chi) = \bar{R}(t-\tau;x-\chi)$$

for all $t,\ \tau \in [0,\infty)$, $x,\ \chi \in \mathbb{R}$ and some function $\bar{R}$. If u is wide-sense stationary in the temporal direction but wide-sense cyclostationary in the spatial direction, then

$$R^u(t,\tau;x+\mathrm{X}m,\chi+\mathrm{X}m) = \tilde{R}(t-\tau;x,\chi)$$

for all $t,\ \tau \in [0,\infty)$, $x,\ \chi \in \mathbb{R}$, $m \in \mathbb{Z}$ and some function $\tilde{R}$. In this paper we abuse notation by writing $R^u(t-\tau;x,\chi)$ instead of $\tilde{R}(t-\tau;x,\chi)$. The value $R^u(t,t;x,x) = R^u(0;x,x)$ is the variance of u at spatial location x.

IV. Frequency Representation of Periodic Operators

In this section we review the representation of spatially periodic operators in the Fourier (frequency) domain. We show that it is possible to exploit the particular structure of spatially periodic operators in the frequency domain to obtain a matrix representation of these operators that lends itself more easily to analysis and numerical computations.

Let $\hat{u}$ and $\hat{y}$ denote the Fourier transforms of two spatial functions u and y, respectively. If u and y are related by (6) then their Fourier transforms $\hat{u}$ and $\hat{y}$ satisfy

$$\hat{y}(k_x) = \int_{\mathbb{R}} \hat{G}(k_x,\kappa)\hat{u}(\kappa)d\kappa \tag{9}$$

where the kernel functions G and $\hat{G}$ may contain distributions in general, see Fig. 1(a).

It is a standard fact that if the operator G is a spatially invariant operator (i.e., if it commutes with all spatial shifts) then its representation in the Fourier domain is a multiplication operator [5], [18], that is, there exists a function $\hat{g}(k_x)$ such that

$$\hat{y}(k_x) = \hat{g}(k_x)\hat{u}(k_x). \tag{10}$$

This means that the kernel $\hat{G}(k_x,\kappa)$ in (9) can be represented by

$$\hat{G}(k_x,\kappa) = \hat{g}(k_x)\delta(k_x-\kappa).$$

In other words, spatially invariant operators have Fourier kernel functions $\hat{G}(k_x,\kappa)$ that are "diagonal", i.e., they are a function of only $k_x-\kappa$. This can be visualized as an "impulse sheet" along the diagonal $k_x = \kappa$ whose strength is given by the function $\hat{g}(k_x)$.

We now investigate the structure of the kernel functions of spatially periodic operators. Consider a spatially periodic multiplication operator with period $\mathrm{X} = 2\pi/\Omega$ defined by

$$y(x) = e^{j\Omega l x}u(x), \quad l \in \mathbb{Z}.$$

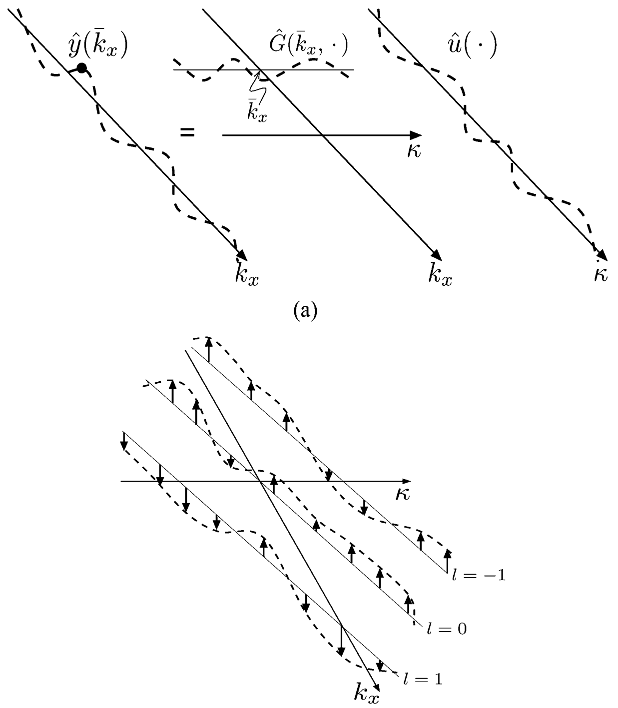


Fig. 1. (a) Schematic representation of $\hat{y}(\bar{k}_x) = \int \hat{G}(\bar{k}_x, \kappa)\hat{u}(\kappa)d\kappa$, to be thought of as a generalization of matrix-vector multiplication. (b) The frequency domain kernel representation $\hat{G}$ of a spatially periodic operator. The kernel is made of an array of equally-spaced diagonal impulse sheets.

From the standard shift property of the Fourier transform we have

$$\hat{y}(k_x) = \hat{u}(k_x - \Omega l)$$

i.e., $\hat{y}$ is a shifted version of $\hat{u}$. Such shifts are represented in (9) by kernel functions of the form

$$\hat{G}(k_x, \kappa) = \delta(k_x - \kappa - \Omega l).$$

This can be visualized as an impulse sheet of constant strength along the subdiagonal $k_x - \kappa = \Omega l$.

Now consider multiplication by a general spatially periodic function f of period $\mathrm{X} = 2\pi/\Omega$. Let f_l be the Fourier series coefficients of f,

$$f(x) = \sum_{l \in \mathbb{Z}} f_l e^{j\Omega l x}.$$

Using the above series, the shift property of the Fourier transform, and the linearity of multiplication operators, we conclude that

$$y(x) = f(x)u(x) \iff \hat{y}(k_x) = \sum_{l \in \mathbb{Z}} f_l \hat{u}(k_x - \Omega l)$$

i.e., $\hat{y}$ is the sum of weighted shifts of $\hat{u}$. Thus the kernel function of a periodic pure multiplication operator is of the form

$$\hat{G}(k_x, \kappa) = \sum_{l \in \mathbb{Z}} f_l \delta(k_x - \kappa - \Omega l)$$

which converges in the sense of distributions. This can be visualized as an array of diagonal impulse sheets at $k_x - \kappa = \Omega l$ with relative strength given by f_l, where f_l is the lth Fourier series coefficient of the function $f(x)$.

Let us now find the structure of a general spatially periodic operator. The cascade of a pure multiplication by $e^{j\Omega l x}$ followed by a spatially invariant operator with Fourier symbol $\hat{g}(k_x)$ has a kernel function given by

$$\hat{g}(k_x)\delta(k_x - \kappa - \Omega l).$$

It is easy to see that sums and cascades of such basic periodic operators produce an operator with a kernel function (in the space of distributions) of the form

$$\hat{G}(k_x, \kappa) = \sum_{l \in \mathbb{Z}} \hat{g}_l(k_x)\delta(k_x - \kappa - \Omega l) \tag{11}$$

where $\hat{g}_l(k_x)$, for each k_x, can in general be a matrix. Such a kernel function can be visualized in Fig. 1(b). It is not difficult to show that (11) describes the most general form of a spatially periodic operator [19].

Consider spatially periodic operators with kernel functions described by (11). These operators are completely specified by the family of (matrix-valued) functions $\{\hat{g}_l(k_x)\}_{l \in \mathbb{Z}}$. We further assume that $\hat{g}_l(\cdot)$, $l \in \mathbb{Z}$, are continuous functions of k_x. It is interesting to observe certain special subclasses of these operators:

1) A *spatially invariant* operator has a kernel function of the form (11) in which $\hat{g}_l = 0$ for $l \neq 0$ (it is purely "diagonal").
2) A *periodic pure multiplication* operator has a kernel function of the form (11) in which all the functions $\hat{g}_l$ are constant in their arguments (it is a "Toeplitz" operator).

We next show how the special structure of (11) can be exploited to yield a new representation for $\hat{G}$. Consider (9) and rewrite $k_x \in \mathbb{R}$ as $\theta + \Omega n$ for some $n \in \mathbb{Z}$ and $\theta \in [0, \Omega)$. Then using (11) we have

$$\begin{aligned} \hat{y}(\theta + \Omega n) &= \int_{\mathbb{R}} \hat{G}(\theta + \Omega n, \kappa)\, \hat{u}(\kappa)\, d\kappa \\ &= \sum_{l \in \mathbb{Z}} \hat{g}_l(\theta + \Omega n)\, \hat{u}(\theta + \Omega n - \Omega l) \\ &= \sum_{m \in \mathbb{Z}} \hat{g}_{n-m}(\theta + \Omega n)\, \hat{u}(\theta + \Omega m). \end{aligned} \tag{12}$$

Forming the bi-infinite vectors $y_\theta := \{\hat{y}(\theta + \Omega n)\}_{n \in \mathbb{Z}}$ and $u_\theta := \{\hat{u}(\theta + \Omega m)\}_{m \in \mathbb{Z}}$, (12) becomes

$$\begin{bmatrix} \vdots \\ \hat{y}(\theta-\Omega) \\ \hat{y}(\theta) \\ \hat{y}(\theta+\Omega) \\ \vdots \end{bmatrix} = \begin{bmatrix} \ddots & \vdots & \vdots & \vdots & \iddots \\ \cdots & \hat{g}_0(\theta-\Omega) & \hat{g}_{-1}(\theta-\Omega) & \hat{g}_{-2}(\theta-\Omega) & \cdots \\ \cdots & \hat{g}_1(\theta) & \hat{g}_0(\theta) & \hat{g}_{-1}(\theta) & \cdots \\ \cdots & \hat{g}_2(\theta+\Omega) & \hat{g}_1(\theta+\Omega) & \hat{g}_0(\theta+\Omega) & \cdots \\ \iddots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} \vdots \\ \hat{u}(\theta-\Omega) \\ \hat{u}(\theta) \\ \hat{u}(\theta+\Omega) \\ \vdots \end{bmatrix} \tag{13}$$

which we simply denote by

$$y_\theta = \mathcal{G}_\theta u_\theta. \tag{14}$$

As θ varies in $[0, \Omega)$ the bi-infinite matrix $\mathcal{G}_\theta$ fully describes the kernel $\hat{G}$. We will henceforth refer to the representation (13) as the *lifted* representation [12], [13]. In this setting, the special subclasses of operators mentioned previously have particularly simple forms:

1) Spatially invariant operators have the *diagonal* representation

$$\mathcal{G}_\theta = \mathrm{diag}\{\hat{g}(\theta + \Omega n)\}_{n\in\mathbb{Z}}. \tag{15}$$

For example, if $G = \partial_x$ then the Fourier symbol of G is jk_x and $\mathcal{G}_\theta$ is given by (15) with $\hat{g}(k_x) = jk_x$.

2) Periodic pure multiplication operators have the *Toeplitz* representation

$$\mathcal{G}_\theta = \mathrm{toep}\{\ldots, f_2, f_1, \boxed{f_0}, f_{-1}, f_{-2}, \ldots\} \tag{16}$$

where the boxed element corresponds to the main diagonal of $\mathcal{G}_\theta$. Notice that $\mathcal{G}_\theta$ is independent of θ. For example, if $f(x) = \cos(\Omega x) = (e^{j\Omega x} + e^{-j\Omega x})/2$ then $\mathcal{G}_\theta$ is given by (16) with $f_1 = f_{-1} = 1/2$ and $f_l = 0$, $l \neq \pm 1$.

Remark 1: A way to interpret the matrix representation introduced above is to think of $\mathcal{G}_\theta$, for every given θ, as being composed of equally-spaced "samples" of the impulse sheets of $\hat{G}$. As θ changes in $[0, \Omega)$, this sampling grid slides on the impulse sheets of $\hat{G}$, picking up the values $\hat{g}_l(\theta + \Omega n)$, $l,\ n \in \mathbb{Z}$ that constitute the elements of $\mathcal{G}_\theta$. ■

Remark 2: Let $\mathcal{M}_\theta$ denote the unitary lifting operator in the frequency domain [13]. Then $u_\theta = \mathcal{M}_\theta \hat{u}$, $y_\theta = \mathcal{M}_\theta \hat{y}$, and $\mathcal{G}_\theta = \mathcal{M}_\theta \hat{G} \mathcal{M}_\theta^*$. Furthermore $\mathcal{M}_\theta$, $\theta \in [0, \Omega)$ is unitary and therefore preserves norms. To see this, suppose $\hat{u}$ belongs to a subset D of $L^2(\mathbb{R})$ such that

$$\hat{u} \in \mathsf{D} \subset L^2(\mathbb{R}) \implies u_\theta \in \ell^2(\mathbb{Z}) \text{ for every } \theta \in [0, \Omega).$$

Then

$$\|\hat{u}\|_{L^2}^2 = \int_{\mathbb{R}} \|\hat{u}(k_x)\|^2 dk_x = \sum_{m\in\mathbb{Z}} \int_0^\Omega \|\hat{u}(\theta + \Omega m)\|^2 d\theta = \int_0^\Omega \|u_\theta\|_{\ell^2}^2 d\theta$$

where $\|\cdot\|$ is the standard Euclidean norm. We also have

$$\sum_{l\in\mathbb{Z}} \int_{\mathbb{R}} \mathrm{trace}\,[\hat{g}_l(k_x)\hat{g}_l^*(k_x)]dk_x = \int_0^\Omega \mathrm{trace}\,[\mathcal{G}_\theta \mathcal{G}_\theta{}^*]d\theta = \int_0^\Omega \|\mathcal{G}_\theta\|_{\mathrm{HS}}^2\, d\theta \tag{17}$$

with $\|T\|_{\mathrm{HS}}^2 := \mathrm{trace}\ [TT^*]$ being the square of the Hilbert–Schmidt norm[1] of T. ■

V. Representations of Spatially Periodic Systems

In this section we use the lifting transform introduced in the previous section to represent a general spatially periodic LTI system as a family of decoupled bi-infinite matrix-valued LTI systems.

Consider the spatially periodic LTI system G described by

$$\begin{aligned}[\partial_t \psi](t,x) &= [A\psi](t,x) + [Bu](t,x) \\ y(t,x) &= [C\psi](t,x) + [Du](t,x)\end{aligned} \tag{18}$$

where x belongs to $\mathbb{R}$ and the operators A, B, C, and D are spatially periodic[2] with common period X. The operator A is defined on a dense domain $\mathsf{D}(A) \subset L^2(\mathbb{R})$, is closed, and generates a *strongly continuous* semigroup (also known as C_0-semigroup) denoted by e^{At} [4]. The operators B, C, and D are bounded. The functions u, y, and ψ are the spatio-temporal input, output, and state of the system, respectively, and $\psi(t,\cdot) \in L^2(\mathbb{R})$ for $t \geq 0$. We refer to A as the *infinitesimal generator* of the system.

Example 1: An example of (18) is the spatially periodic heat equation on the real line

$$\begin{aligned}\partial_t \psi(t,x) &= \left(\partial_x^2 - \alpha\cos(\Omega x)\right)\psi(t,x) + u(t,x) \\ y(t,x) &= \psi(t,x)\end{aligned} \tag{19}$$

with real $\alpha \neq 0$ and $\Omega > 0$. Here $A = \partial_x^2 - \alpha\cos(\Omega x)$ with domain

$$\mathsf{D}(A) = \left\{\phi \in L^2(\mathbb{R}) \mid \phi, \frac{d\phi}{dx} \text{ absolutely continuous}, \frac{d^2\phi}{dx^2} \in L^2(\mathbb{R})\right\}$$

B and C are the identity operators on $L^2(\mathbb{R})$, and $D = 0$. ■

System (18) can also be represented by a spatio-temporal kernel G,

$$y(t,x) = \left(Gu\right)(t,x) = \int_{\mathbb{R}}\int_0^\infty G(t,\tau;x,\chi)u(\tau,\chi)d\tau d\chi$$

with

$$G(t,\tau;x + \mathrm{X}m, \chi + \mathrm{X}m) = G(t-\tau;x,\chi) \tag{20}$$

for all $t \geq \tau \geq 0$, $x, \chi \in \mathbb{R}$, and $m \in \mathbb{Z}$. It can be shown that if $\psi(0,\cdot) \in L^2(\mathbb{R})$ and $\|u(t,\cdot)\|_{L^2}$ is a bounded function of t, then u, y, and ψ are guaranteed to reside in $L^2(\mathbb{R})$ for every $t \geq 0$ and we can apply the spatial Fourier transform to both sides of (18) to get

$$\begin{aligned}[\partial_t \hat{\psi}](t,k_x) &= [\hat{A}\hat{\psi}](t,k_x) + [\hat{B}\hat{u}](t,k_x) \\ \hat{y}(t,k_x) &= [\hat{C}\hat{\psi}](t,k_x) + [\hat{D}\hat{u}](t,k_x)\end{aligned} \tag{21}$$

where the spatial-frequency variable k_x belongs to $\mathbb{R}$. Applying the lifting transform to both sides of (21) we have

$$\begin{aligned}\partial_t \psi_\theta(t) &= \mathcal{A}_\theta \psi_\theta(t) + \mathcal{B}_\theta u_\theta(t) \\ y_\theta(t) &= \mathcal{C}_\theta \psi_\theta(t) + \mathcal{D}_\theta u_\theta(t)\end{aligned} \tag{22}$$

where the new frequency variable θ belongs to $[0, \Omega)$. System (22) is now fully decoupled in θ, i.e., the original system (18) is equivalent to the *family* of systems (22) parameterized by $\theta \in [0, \Omega)$. Finally, system (22) has impulse response

$$\mathcal{G}_\theta(t) = \mathcal{C}_\theta e^{\mathcal{A}_\theta t}\mathcal{B}_\theta + \mathcal{D}_\theta \tag{23}$$

[1]The Hilbert–Schmidt norm of an operator is a generalization of the Frobenius norm of finite-dimensional matrices $\|A\|_{\mathrm{F}}^2 = \sum_{m,n} |a_{mn}|^2 = \mathrm{trace}\ [AA^*]$.

[2]Any number of these operators can be spatially invariant, such operators constituting a subclass of spatially periodic operators.

and transfer function

$$\mathcal{G}_\theta(\omega) = \mathcal{C}_\theta\,(j\omega\mathcal{I} - \mathcal{A}_\theta)^{-1}\mathcal{B}_\theta + \mathcal{D}_\theta.$$

Example 1 Continued: Let us return to the example of the periodic heat equation described by (19). Rewriting the system in its lifted representation (22) we have $\mathcal{B}_\theta = \mathcal{C}_\theta = \mathcal{I}$, $\mathcal{D}_\theta = 0$, $\mathcal{A}_\theta$ is given by

$$\mathcal{A}_\theta = -\begin{bmatrix} \ddots & \vdots & \vdots & \vdots & \cdot^{\cdot^{\cdot}} \\ \cdots & (\theta-\Omega)^2 & \frac{\alpha}{2} & 0 & \cdots \\ \cdots & \frac{\alpha}{2} & \theta^2 & \frac{\alpha}{2} & \cdots \\ \cdots & 0 & \frac{\alpha}{2} & (\theta+\Omega)^2 & \cdots \\ \cdot^{\cdot^{\cdot}} & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

and

$$\mathcal{G}_\theta(\omega) = (j\omega\mathcal{I} - \mathcal{A}_\theta)^{-1}.$$

Remark 3: An advantage of the lifted representation is that it allows for (22) to be treated like a multivarible system, under some technical assumptions. It is then possible to extend many existing tools from linear systems theory to (22). For example [20] uses the fact that in general $\mathcal{A}_\theta$ has discrete spectrum for every θ to generalize the Nyquist criterion to determine the stability of (22), and [21] employes Geršgorin-like arguments to locate the spectrum of $\mathcal{A}_\theta$. Also [22], [23] use the lifted representation to investigate the occurrence of parametric resonance in spatially periodic systems. Finally [21] utilizes the bi-infinite matrix structure of the lifted representation to perform a perturbation analysis of the $\mathcal{H}^2$ norm.

Remark 4: It is often useful to think of a spatially periodic system as the feedback interconnection of a spatially invariant system and a spatially periodic operator [19]. For example, system (19) can be rewritten as the feedback interconnection of the spatially invariant system G^{o} described by

$$\begin{aligned} \partial_t\psi(t,x) &= \partial_x^2\psi(t,x) + w(t,x) \\ y(t,x) &= \psi(t,x) \end{aligned}$$

and the spatially periodic pure multiplication operator $F(x) = -\alpha\cos(\Omega x)$,

$$w(t,x) = -\alpha\cos(\Omega x)y(t,x) + u(t,x)$$

as in Fig. 2(a). More generally, a wide class of spatially periodic systems (18) can be written as the linear fractional transformation (LFT) [24] of a spatially periodic system G^{o} with *spatially invariant infinitesimal generator* A^{o} and a bounded spatially periodic operator F, i.e.,

$$\begin{aligned} \partial_t\psi(t,x) &= \big(A^{\mathrm{o}} + B^{\mathrm{o}}FC^{\mathrm{o}}\big)\psi(t,x) + Bu(t,x) \\ y(t,x) &= C\psi(t,x) + Du(t,x) \end{aligned} \quad (24)$$

as shown in Fig. 2(b). The (possibly unbounded) operators A^{o}, B^{o}, C^{o} are spatially invariant and the bounded operators B, C, D, F are spatially periodic. ■

Stability Conditions and the Spectrum

We next briefly summarize some basic facts regarding the spectrum of spatially periodic operators and the exponential stability of spatially periodic LTI systems.

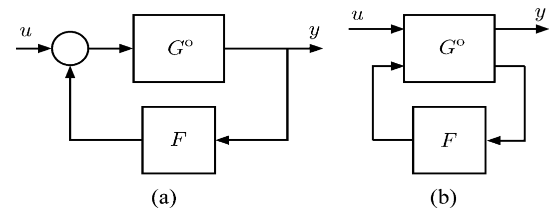


Fig. 2. (a) A spatially periodic system, represented as the closed-loop interconnection of spatially invariant system G^{o} and spatially periodic operator F. (b) A spatially periodic system can be represented as the LFT of system G^{o} with spatially invariant dynamics and spatially periodic operator F.

Since the Fourier and lifting transformations are both unitary and the spectral properties of operators remain preserved under unitary transformations, we have

$$\Sigma(A) = \Sigma(\hat{A}) = \overline{\bigcup_{\theta\in[0,\Omega)} \Sigma(\mathcal{A}_\theta)} = \bigcup_{\theta\in[0,\Omega]} \Sigma(\mathcal{A}_\theta) \quad (25)$$

with the last equality being valid under the assumption of continuous dependence of $\mathcal{A}_\theta$ on θ. In the case where A is spatially invariant and thus $\mathcal{A}_\theta = \mathrm{diag}\{\ldots, \hat{A}_0(\theta+\Omega n), \ldots\}$, (25) simplifies to $\Sigma(A) = \overline{\bigcup_{k_x\in\mathbb{R}} \Sigma_{\mathrm{p}}\big(\hat{A}_0(k_x)\big)}$.

A strongly continuous semigroup on a Hilbert space is exponentially stable if there exist positive constants M and β such that $\|e^{At}\| \leq Me^{-\beta t}$ for $t \geq 0$ [4]. The temporal growth properties of spatially distributed systems are preserved under Fourier and lifting transformations and thus (18) is exponentially stable if and only if (22) is exponentially stable for all values of the parameter θ.

VI. $\mathcal{H}^2$ Norm of Spatially Periodic Systems

In this section we introduce the notion of $\mathcal{H}^2$ norm for spatially periodic LTI *systems* and give both deterministic and stochastic interpretations of this norm. To this end, we first examine a similar norm, which we denote by $\mathcal{H}^2_{\mathrm{sp}}$, for the case of spatially periodic *operators*. We demonstrate that the lifted representation (22) plays a central role in allowing us to extend tools from standard linear systems theory to the analysis of spatially periodic systems. Furthermore, we use the lifted representation to investigate numerical approximations in computing the $\mathcal{H}^2$ norm. Due to space limitations we do not discuss the $\mathcal{H}^\infty$ norm of spatially periodic systems here and instead refer the interested reader to [19]. Throughout this section we assume that spatially periodic systems are exponentially stable.

A. Deterministic Interpretation of the $\mathcal{H}^2$ Norm

1) Spatially Periodic Operators: The $\mathcal{H}^2$ norm of a linear time-periodic system was defined in [25]. Here we employ a similar approach to define the $\mathcal{H}^2_{\mathrm{sp}}$ norm of a spatially periodic operator.

Let us consider a scalar-valued spatially periodic operator with kernel G that satisfies $G(x,\chi) = G(x + \mathrm{X}m, \chi + \mathrm{X}m)$, $m \in \mathbb{Z}$. Note that $G(\cdot,\chi)$ is the output of G when the input $v^\chi(x) := \delta(x-\chi)$ is a spatial impulse at $x = \chi$. Since G is not spatially invariant the output corresponding to the input v^{χ_1} is in general different from the output corresponding to the input v^{χ_2} if $\chi_1 \neq \chi_2$. This is true unless χ_1 and χ_2 satisfy $\chi_2 = \chi_1 + \mathrm{X}m$, $m \in \mathbb{Z}$, in which case the output corresponding to the second input will be merely an $\mathrm{X}m$-shifted version of the output corresponding to the first input. This means that to fully capture the effect of the spatially periodic operator, one needs to evaluate all its outputs corresponding to impulsive inputs applied at every point inside one spatial period X. Therefore we define the $\mathcal{H}^2_{\text{sp}}$ norm of a spatially periodic operator G as

$$\|G\|^2_{\mathcal{H}^2_{\text{sp}}} := \frac{1}{\mathrm{X}} \int_0^{\mathrm{X}} \left[\int_{\mathbb{R}} \left| \left(G v^{\chi} \right)(x) \right|^2 dx \right] d\chi. \quad (26)$$

We next provide a characterization of this norm in the frequency domain.

Theorem 1: Let G be a scalar-valued spatially periodic operator with period $\mathrm{X} = 2\pi/\Omega$. Then

$$\|G\|^2_{\mathcal{H}^2_{\text{sp}}} = \frac{1}{2\pi} \int_0^{\Omega} \|\mathcal{G}_\theta\|^2_{\text{HS}} d\theta \quad (27)$$

where $\|\mathcal{G}_\theta\|^2_{\text{HS}} := \text{trace}\, [\mathcal{G}_\theta \mathcal{G}_\theta{}^*]$.

Proof: See Appendix I-B . ■

2) Spatially Periodic LTI Systems: Consider a scalar-valued spatially periodic LTI system described by (18) whose kernel $G(t-\tau; x, \chi)$ satisfies (20). Let us assume for now that $D = 0$. Based on an argument similar to the one presented in the previous paragraph and using the time invariance of G we define

$$\|G\|^2_{\mathcal{H}^2} := \frac{1}{\mathrm{X}} \int_0^{\mathrm{X}} \int_{\mathbb{R}} \int_0^{\infty} \left| \left(G u^{\chi} \right)(t, x) \right|^2 dt\, dx\, d\chi$$

where $u^{\chi}(t, x) := \delta(t)\delta(x - \chi)$.

Notation: We shall denote by $\underline{G}$ the system described by (18) without the feed-through operator D, i.e., $\underline{G} = G - D$. ■

For an exponentially stable multi-input multi-output (MIMO) spatially periodic LTI system G with $D \neq 0$, the appropriate definition of the $\mathcal{H}^2$ norm is

$$\begin{aligned} \|G\|^2_{\mathcal{H}^2} &:= \| \underline{G} u\|^2_{\mathcal{H}^2} + \|D\|^2_{\mathcal{H}^2_{\text{sp}}} \\ &= \frac{1}{\mathrm{X}} \int_0^{\mathrm{X}} \int_{\mathbb{R}} \int_0^{\infty} \text{trace}\, [\underline{G}(t; x, \chi) \underline{G}^*(t; \chi, x)] dt\, dx\, d\chi + \|D\|^2_{\mathcal{H}^2_{\text{sp}}}. \end{aligned} \quad (28)$$

The next theorem provides a procedure for computing the $\mathcal{H}^2$ norm in the spatial-frequency domain using (operator-valued) algebraic Lyapunov equations.

Theorem 2: Let G be an exponentially stable spatially periodic system for which $\|G\|^2_{\mathcal{H}^2}$ as defined in (28) is finite. Then

$$\|G\|^2_{\mathcal{H}^2} = \frac{1}{4\pi^2} \int_0^{\Omega} \int_{\mathbb{R}} \|\underline{\mathcal{G}}_\theta(\omega)\|^2_{\text{HS}}\, d\omega\, d\theta + \|D\|^2_{\mathcal{H}^2_{\text{sp}}}$$

where $\underline{\mathcal{G}}_\theta(t) := \mathcal{C}_\theta e^{\mathcal{A}_\theta t} \mathcal{B}_\theta$ and $\|D\|^2_{\mathcal{H}^2_{\text{sp}}} = (1/2\pi) \int_0^{\Omega} \|\mathcal{D}_\theta\|^2_{\text{HS}} d\theta$ as given by Theorem 1. Furthermore

$$\begin{aligned} \|G\|^2_{\mathcal{H}^2} &= \frac{1}{2\pi} \int_0^{\Omega} \text{trace}\, [\mathcal{C}_\theta \mathcal{P}_\theta \mathcal{C}^*_\theta] d\theta + \|D\|^2_{\mathcal{H}^2_{\text{sp}}} \\ &= \frac{1}{2\pi} \int_0^{\Omega} \text{trace}\, [\mathcal{B}^*_\theta \mathcal{Q}_\theta \mathcal{B}_\theta] d\theta + \|D\|^2_{\mathcal{H}^2_{\text{sp}}} \end{aligned}$$

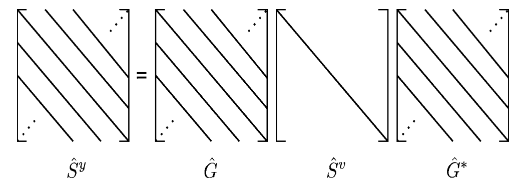


Fig. 3. A spatially periodic operator G with wide-sense stationary input v has a wide-sense *cyclostationary* output y. The diagonal lines depict the support of impulse sheets in the frequency domain kernel representation.

where $\mathcal{P}_\theta$ and $\mathcal{Q}_\theta$ are solutions of the θ-parameterized algebraic Lyapunov equations

$$\mathcal{A}_\theta \mathcal{P}_\theta + \mathcal{P}_\theta \mathcal{A}^*_\theta = -\mathcal{B}_\theta \mathcal{B}^*_\theta,\; \mathcal{A}^*_\theta \mathcal{Q}_\theta + \mathcal{Q}_\theta \mathcal{A}_\theta = -\mathcal{C}^*_\theta \mathcal{C}_\theta. \quad (29)$$

Proof: See Appendix I-C. ■

In practice, to calculate the $\mathcal{H}^2$ norm of a spatially periodic system one has to perform a finite-dimensional truncation of the bi-infinite matrices $\mathcal{A}_\theta$, $\mathcal{B}_\theta$, $\mathcal{C}_\theta$, and $\mathcal{D}_\theta$. In Section VI-C we investigate the convergence properties of such truncations.

B. Stochastic Interpretation of the $\mathcal{H}^2$ Norm

The stochastic interpretation of the $\mathcal{H}^2$ norm for spatially periodic operators and LTI systems uses the notion of cylcostationary random fields and spectral-correlation density operators. These are briefly discussed in Appendix I-A.

1) Spatially Periodic Operators: Consider a scalar-valued spatially periodic operator with Fourier kernel $\hat{G}(k_x, \kappa) = \sum_{l \in \mathbb{Z}} \hat{G}_l(k_x) \delta(k_x - \kappa - \Omega l)$. Assume that the input to G is a wide-sense stationary spatial random field v with autocorrelation $R^v(x, \chi) = R^v(x - \chi)$ and spectral-correlation density $\hat{S}^v(k_x, \kappa) = \hat{S}^v_0(k_x)\delta(k_x - \kappa)$, where $\hat{S}^v_0(\cdot)$ is the spectral density of v. Then, as shown in Appendix I-A, the output $y(x) = \big(Gv\big)(x)$ has the spectral-correlation density

$$\hat{S}^y = \hat{G} \hat{S}^v \hat{G}^*$$

which can be visualized as in Fig. 3. The operator $\hat{S}^y$ inherits the structure of $\hat{G}$ and therefore y becomes a *cyclostationary* random field. More specifically,

$$\begin{aligned} \hat{S}^y(k_x, \kappa) &= \big(\hat{G}\hat{S}^v\hat{G}^*\big)(k_x, \kappa) \\ &=: \sum_{l \in \mathbb{Z}} \hat{S}^y_l(k_x) \delta(k_x - \kappa - \Omega l) \end{aligned} \quad (30)$$

where

$$\hat{S}^y_0(k_x) = \sum_{l \in \mathbb{Z}} |\hat{G}_l(k_x)|^2 \hat{S}^v_0(k_x). \quad (31)$$

All other $\hat{S}^y_l(\cdot)$, $0 \neq l \in \mathbb{Z}$, can be found by forming the composition (30) but are not relevant to the discussion here. The autocorrelation R^y, which is the inverse Fourier transform of $\hat{S}^y$, satisfies

$$R^y(x + \mathrm{X}m, \chi + \mathrm{X}m) = R^y(x, \chi)$$

for all $m \in \mathbb{Z}$, and we have the following theorem.

Theorem 3: Consider the scalar spatially periodic operator G. If $y = Gu$ and u is a white random field, then

$$\|G\|^2_{\mathcal{H}^2_{\rm sp}} = \frac{1}{\rm X}\int_0^{\rm X} R^y(x,x)dx = \frac{1}{2\pi}\int_{\mathbb{R}} \hat{S}^y_0(k_x)dk_x.$$

Proof: See Appendix I-D. ■

Thus the $\mathcal{H}^2_{\rm sp}$ norm of a spatially periodic operator has the stochastic interpretation of being the average over one period of the variance of the output random field when the input is a white random field.

2) Spatially Periodic LTI Systems: Consider a scalar-valued spatially periodic LTI system described by (18) whose kernel $G(t-\tau;x,\chi)$ satisfies (20). Let us assume for now that $D=0$. Suppose that the system G is given an input $u(t,x)$ which is a wide-sense stationary random field in both the temporal and the spatial directions with autocorrelation function that satisfies

$$R^u(t,\tau;x,\chi) = R^u(t-\tau;x-\chi)$$

for all $t \geq \tau \geq 0$ and $x,\chi \in \mathbb{R}$. Therefore, from the discussion presented in the previous paragraph, it follows that the output $y(t,x) = \big(Gu\big)(t,x)$ is a random field with autocorrelation $R^y(t,\tau;x,\chi)$ that satisfies

$$R^y(t,\tau;x+{\rm X}m,\chi+{\rm X}m) = R^y(t-\tau;x,\chi)$$

for all $t \geq \tau \geq 0$, $x,\chi \in \mathbb{R}$, and $m \in \mathbb{Z}$. Namely, the output random field remains wide-sense stationary in the temporal direction due to the time-invariance of G, but inherits the spatial-periodicity of G and thus becomes a cyclostationary random field in the spatial direction. In particular, if input u is a white random field in both the spatial and temporal directions, $\hat{S}^u(\omega,\nu;k_x,\kappa) = \delta(\omega-\nu)\delta(k_x-\kappa)$, then

$$\|G\|^2_{\mathcal{H}^2} = \frac{1}{\rm X}\int_0^{\rm X} R^y(0;x,x)\,dx.$$

Notation: We shall denote by $\underline{R}^y$ the output autocorrelation of system $\underline{G} = G - D$ subject to an input with autocorrelation R^u. ■

For an exponentially stable MIMO spatially periodic LTI system G with $D \neq 0$, the stochastic definition of the $\mathcal{H}^2$ norm is

$$\begin{aligned}\|G\|^2_{\mathcal{H}^2} &= \|\underline{G}\|^2_{\mathcal{H}^2} + \|D\|^2_{\mathcal{H}^2_{\rm sp}}\\ &= \frac{1}{\rm X}\int_0^{\rm X} {\rm trace}\,[\underline{R}^y(0;x,x)]\,dx + \|D\|^2_{\mathcal{H}^2_{\rm sp}}\\ &= \frac{1}{4\pi^2}\int_{\mathbb{R}}\int_{\mathbb{R}} {\rm trace}\,[\hat{\underline{S}}^y_0(\omega;k_x)]\,dk_x\,d\omega + \|D\|^2_{\mathcal{H}^2_{\rm sp}}\end{aligned}$$

where $\hat{\underline{S}}^y$ is the spatio-temporal Fourier transform of $\underline{R}^y$.

C. Numerical Implementation and Finite-Dimensional Truncations

In order to numerically calculate the $\mathcal{H}^2$ norm of a spatially periodic LTI system, one has to perform a finite-dimensional truncation of the infinite-dimensional operators that describe the system. To calculate the $\mathcal{H}^2$ norm using the Lyapunov equations in Theorem 2, a truncation of the operators $\mathcal{A}_\theta$, $\mathcal{B}_\theta$, and $\mathcal{C}_\theta$ has to be performed. It is then necessary to verify that the $\mathcal{H}^2$ norm of the original system is approximated arbitrarily-well by increasing the truncation size.

Notation: We use $\Pi T \Pi$ to denote the $(2N+1)\times(2N+1)$ truncation of the bi-infinite matrix representation of an operator T on ℓ^2, where Π is the projection operator defined by

$${\rm diag}\Big\{\ldots,0,\underbrace{I,\ldots,I,\ldots,I}_{2N+1 \text{ times}},0,\ldots\Big\}.$$

Here I is the identity matrix with dimension equal to that of the elements of T. The operator Π commutes with all (block) diagonal operators, i.e., $T\Pi = \Pi T = \Pi T \Pi$ for diagonal T. Furthermore, $\Pi\Pi = \Pi$ and $(\mathcal{I}-\Pi)\Pi = \Pi(\mathcal{I}-\Pi) = 0$. The following notational conventions may seem rather tedious but the main idea is very simple. The operator $\mathfrak{B}_\theta := \Pi\mathcal{B}_\theta\Pi$ denotes the bi-infinite matrix made from $\mathcal{B}_\theta$ by replacing all elements outside the center $(2N+1)\times(2N+1)$ block with zeros. The operator $\mathtt{B}_\theta := \Pi\mathcal{B}_\theta\Pi|_{\Pi\ell^2}$, the restriction of $\mathcal{B}_\theta$ to the subspace $\Pi\ell^2$, denotes the finite matrix equal to the center $(2N+1)\times(2N+1)$ block of $\mathcal{B}_\theta$, or equivalently, the center $(2N+1)\times(2N+1)$ block of $\mathfrak{B}_\theta$. Thus

$$\mathcal{B}_\theta = \begin{bmatrix} \ddots & \vdots & \iddots \\ \cdots & \mathtt{B}_\theta & \cdots \\ \iddots & \vdots & \ddots \end{bmatrix}, \quad \mathfrak{B}_\theta = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \mathtt{B}_\theta & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

where $\mathcal{B}_\theta$ and $\mathfrak{B}_\theta$ are infinite-dimensional matrices acting on ℓ^2, and $\mathtt{B}_\theta$ is a *finite*-dimensional matrix. The same notational conventions apply to $\mathfrak{C}_\theta$, $\mathfrak{A}^\circ_\theta$, $\mathfrak{B}^\circ_\theta$, $\mathfrak{C}^\circ_\theta$, $\mathfrak{F}$, and $\mathtt{C}_\theta$, $\mathtt{A}^\circ_\theta$, $\mathtt{B}^\circ_\theta$, $\mathtt{C}^\circ_\theta$, $\mathtt{F}$. In addition, $\mathcal{G}^\circ_\theta(\omega) := \mathcal{C}_\theta(j\omega\mathcal{I}-\mathcal{A}^\circ_\theta)^{-1}\mathcal{B}_\theta$, and we recall that $\mathcal{A}_\theta := \mathcal{A}^\circ_\theta + \mathcal{B}^\circ_\theta\mathcal{F}\mathcal{C}^\circ_\theta$, $\mathcal{G}_\theta(\omega) := \mathcal{C}_\theta(j\omega\mathcal{I}-\mathcal{A}_\theta)^{-1}\mathcal{B}_\theta$. ■

For a Hilbert–Schmidt operator $T \in \mathsf{B}_2(\ell^2)$ with matrix representation $\{t_{mn}\}_{m,n\in\mathbb{Z}}$, we have [26]

$$\sum_{m,n=-\infty}^{\infty} |t_{mn}|^2 < \infty.$$

In particular, this means that $\|T-\Pi T\|_{\rm HS}$, $\|T-T\Pi\|_{\rm HS}$, and $\|T-\Pi T\Pi\|_{\rm HS}$ all approach zero as N grows large, where $\|T\|^2_{\rm HS} := {\rm trace}\;[TT^*]$. In this paragraph and in the proof of Theorem 4 below we make use of the fact that Hilbert–Schmidt operators form a two-sided ideal in the algebra of bounded operators on ℓ^2 [16]. Namely, if $T_0, T_1 \in \mathsf{B}(\ell^2)$ and $T_2 \in \mathsf{B}_2(\ell^2)$, then $T_0T_2T_1 \in \mathsf{B}_2(\ell^2)$ and $\|T_0T_2T_1\|_{\rm HS} \leq \|T_0\|_{\ell^2/\ell^2}\|T_2\|_{\rm HS}\|T_1\|_{\ell^2/\ell^2}$.

We assume that the spatially periodic system G can be written in the form described by (24) and depicted in Fig. 2(b), i.e., G is a system resulting from the LFT of a spatially periodic system G° with spatially invariant infinitesimal generator A° and a bounded spatially periodic operator F. We assume that B° and C° are bounded operators, $D=0$, and that both G and G° are exponentially stable and have finite $\mathcal{H}^2$ norm. The latter assumption implies that $\rho(A)$ and $\rho(A^\circ)$ both contain $\mathbb{C}^+$, and in particular, the imaginary axis. We further assume

$$\int_{\mathbb{R}}\int_{\mathbb{R}} \|\big(j\omega I - \hat{A}^\circ(k_x)\big)^{-1}\|^2_{\rm F}\,dk_x\,d\omega < \infty.$$

Then it can be shown [19] that the operators $(j\omega\mathcal{I}-\mathcal{A}^\circ_\theta)^{-1}$, $(j\omega\mathcal{I}-\mathcal{A}_\theta)^{-1}$, and $\mathcal{G}_\theta(\omega)$ belong to $\mathsf{B}_2(\ell^2)$ for all $\omega \in \mathbb{R}$ and

$\theta \in [0, \Omega)$. Since $\mathcal{G}_\theta(\omega)$ is a Hilbert–Schmidt operator it can be approximated to any degree of accuracy by $\Pi \mathcal{G}_\theta(\omega) \Pi$.

Our desired scenario in evaluating norms is to work with the finite-dimensional systems

$$\mathtt{G}_\theta(\omega) := \mathtt{C}_\theta(j\omega \mathtt{I} - [\mathtt{A}_\theta^{\rm o} + \mathtt{B}_\theta^{\rm o}\mathtt{F}\mathtt{C}_\theta^{\rm o}])^{-1}\mathtt{B}_\theta.$$

In particular, we would like to compute the $\mathcal{H}^2$ norm by solving Lyapunov equations involving only the finite-dimensional matrices $\mathtt{A}_\theta$, $\mathtt{B}_\theta$, and $\mathtt{C}_\theta$. The following theorem uses

$$\mathfrak{G}_\theta(\omega) := \mathfrak{C}_\theta(j\omega \mathcal{I} - [\mathcal{A}_\theta^{\rm o} + \mathfrak{B}_\theta^{\rm o}\mathfrak{F}\mathfrak{C}_\theta^{\rm o}])^{-1}\mathfrak{B}_\theta.$$

to establish the connection between $\mathcal{G}_\theta(\omega)$ and $\mathtt{G}_\theta(\omega)$.

Theorem 4: Consider an exponentially stable spatially periodic system described by (24) with finite $\mathcal{H}^2$ norm, and assume that $\int_{\mathbb{R}}\int_{\mathbb{R}} \|(j\omega I - \hat{A}^{\rm o}(k_x))^{-1}\|_{\rm F}^2 dk_x d\omega < \infty$. Then

$$\int_0^\Omega \int_{\mathbb{R}} \|\mathcal{G}_\theta(\omega) - \mathfrak{G}_\theta(\omega)\|_{\rm HS}^2 \, d\omega \, d\theta \to 0$$

as $N \to \infty$. In particular, if $\mathtt{G}_\theta(\omega)$ exists for all $\omega \in \mathbb{R}$ and $\theta \in [0, \Omega)$, then

$$\int_0^\Omega \int_{\mathbb{R}} \|\mathtt{G}_\theta(\omega)\|_{\rm F}^2 \, d\omega \, d\theta = \int_0^\Omega \int_{\mathbb{R}} \|\mathfrak{G}_\theta(\omega)\|_{\rm HS}^2 \, d\omega \, d\theta \to \int_0^\Omega \int_{\mathbb{R}} \|\mathcal{G}_\theta(\omega)\|_{\rm HS}^2 \, d\omega \, d\theta$$

as $N \to \infty$.

Proof: See Appendix I-E. ■

The significance of the above theorem is that the $\mathcal{H}^2$ norm of the spatially periodic system G can now be computed from

$$\|G\|_{\mathcal{H}^2}^2 \approx \frac{1}{2\pi}\int_0^\Omega \text{trace}\,[\mathtt{C}_\theta \mathtt{P}_\theta \mathtt{C}_\theta^*] d\theta = \frac{1}{2\pi}\int_0^\Omega \text{trace}\,[\mathtt{B}_\theta^* \mathtt{Q}_\theta \mathtt{B}_\theta] d\theta$$

where $\mathtt{P}_\theta$ and $\mathtt{Q}_\theta$ are the solutions of the finite-dimensional Lyapunov equations

$$\mathtt{A}_\theta \mathtt{P}_\theta + \mathtt{P}_\theta \mathtt{A}_\theta^* = -\mathtt{B}_\theta \mathtt{B}_\theta^*, \quad \mathtt{A}_\theta^* \mathtt{Q}_\theta + \mathtt{Q}_\theta \mathtt{A}_\theta = -\mathtt{C}_\theta^* \mathtt{C}_\theta.$$

Obviously this is much simpler than solving the infinite-dimensional Lyapunov (29) of Theorem 2.

Remark 5: In many problems it is often needed to take extremely large truncations of the system matrices to capture the important characteristics of the spatially periodic system. This can lead to expensive computations. It is often physically meaningful to regard the spatially periodic operators as additive or multiplicative perturbations of spatially invariant ones. For example, the infinitesimal generator in (24) can often be decomposed as $A = A^{\rm o} + \epsilon E$ where where $A^{\rm o}$ is a spatially invariant operator, E is a spatially periodic operator, and $\epsilon \in \mathbb{C}$ is small. This has prompted a perturbation analysis of the $\mathcal{H}^2$ norm [21]. Using the mentioned perturbation setup, [21] shows that the problem of computing the $\mathcal{H}^2$ norm of a spatially periodic system collapses to solving recursive sequences of *finite-dimensional* Lyapunov equations. These Lyapunov equations involve matrices that are of the same size as the Euclidean dimension of the spatially periodic system (18). The procedure outlined in [21] does not involve truncations and is exact in the sense of perturbations. ■

VII. Examples of PDEs With Spatially Periodic Coefficients

In this section we illustrate how stability properties and system norms of PDEs can be changed by introducing spatially periodic feedback gains. Our results reveal that, depending on the values of amplitude and frequency of the periodic gain, both the stability and the input-output norms of the system can be modified.

A. Linearized Ginzburg–Landau Equation

We return to our example of the linearized Ginzburg–Landau equation considered in Section II. Let $f(x) := \alpha\cos(\Omega x)$, $\alpha \in \mathbb{R}$. While (1)–(5) can be used to analyze the stability of the system, the equation with external excitation

$$\begin{aligned} \partial_t \phi &= z_1\phi + z_2\partial_x^2\phi + z_3(2|f|^2\phi + f^2\phi^*) + u \\ y &= \phi \end{aligned} \tag{32}$$

also allows the characterization of other important system-theoretic properties such as input-output norms. Under the assumption that the forcing u is a real spatio-temporal function and using the new state defined by (3) we can rewrite the system equations in the form described by (24), where

$$A^{\rm o} = \begin{bmatrix} z_1 + |\alpha|^2 z_3 + z_2\partial_x^2 & \frac{1}{2}\alpha^2 z_3 \\ \frac{1}{2}(\alpha^*)^2 z_3^* & z_1^* + |\alpha|^2 z_3^* + z_2^*\partial_x^2 \end{bmatrix}$$

$$B^{\rm o} = C^{\rm o} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \; B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \; C = [1 \quad 0], \; D = 0$$

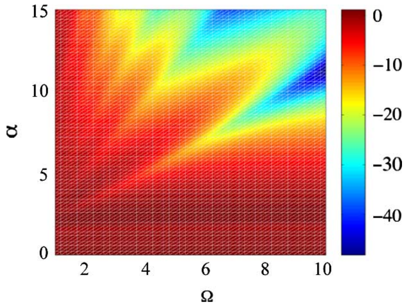
$$F = \frac{1}{2}\begin{bmatrix} |\alpha|^2 z_3 & \frac{1}{2}\alpha^2 z_3 \\ \frac{1}{2}(\alpha^*)^2 z_3^* & |\alpha|^2 z_3^* \end{bmatrix}\cos(2\Omega x).$$

Clearly $A = A^{\rm o} + B^{\rm o}FC^{\rm o}$ is the sum of a spatially invariant and a spatially periodic operator. We use the frequency representation presented in Section IV to perform a numerical approximation of A and investigate the location of its eigenvalues, as required to determine the stability of system (32). We note that technically (32) cannot be interpreted as the linearized GL equation since $\phi_0 = f(x)\exp(j\omega_0 t)$ is no longer a limit cycle solution of the nonlinear GL equation for the choice $f(x) = \alpha\cos(\Omega x)$. Nevertheless, this problem is worth investigating because it represents an example of a system whose stability can be changed by spatially periodic feedback gains.

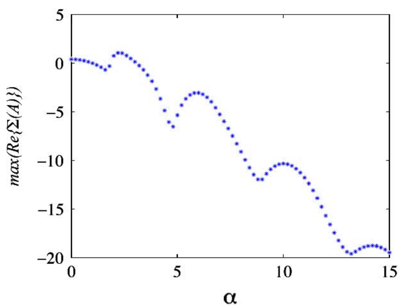
Fig. 4(a) illustrates the maximum real part of $\Sigma(A)$ [which is equal to $\sup_{\theta\in[0,\Omega)}$ of the eigenvalues of the operator $\mathcal{A}_\theta$] as a function of α and Ω for $c_0 = 0.4$, $\rho = 0.4$, and $\omega_0 = 5$; see (2). It can be readily shown that for $\alpha = 0$ we have $\sup_\theta(\max \text{Re}\{\lambda(\mathcal{A}_\theta)\}) = \rho = 0.4$, which means that system (1) is open-loop unstable. Clearly, closing the loop by introducing the spatially periodic operator F in feedback changes the value of $\sup_\theta(\max \text{Re}\{\lambda(\mathcal{A}_\theta)\})$ significantly, as illustrated in Fig. 4(a). Fig. 4(b) shows $\sup_\theta(\max \text{Re}\{\lambda(\mathcal{A}_\theta)\})$ as a function of α for $\Omega \approx 4$.

B. Linearized Swift–Hohenberg Equation

In this example we show that the $\mathcal{H}^2$ norm and stability of a PDE with periodic coefficients can be altered by changing



(a)



(b)

Fig. 4. (a) $\max \mathrm{Re}\{\Sigma(A)\} = \sup_\theta(\max \mathrm{Re}\{\lambda(\mathcal{A}_\theta)\})$ as a function of α and Ω for $f(x) = \alpha\cos(\Omega x)$, $c_0 = 0.4$, $\rho = 0.4$, and $\omega_0 = 5$. (b) $\max \mathrm{Re}\{\Sigma(A)\} = \sup_\theta(\max \mathrm{Re}\{\lambda(\mathcal{A}_\theta)\})$ as a function of α for $f(x) = \alpha\cos(\Omega x)$, $\Omega \approx 4$, $c_0 = 0.4$, $\rho = 0.4$, and $\omega_0 = 5$.

the amplitude and frequency of the periodic term. We consider the Swift–Hohenberg equation which is of interest in hydrodynamics [8], [27], [28] and nonlinear optics [29], [30], as well as other branches of physics [8].

The linearization of the Swift–Hohenberg equation around its time independent spatially periodic solution leads to a PDE with spatially periodic coefficients of the form [9]

$$\begin{aligned} \partial_t\psi &= -(\partial_x^2+1)^2\psi - c\psi + f\psi + u \\ y &= \psi, \end{aligned} \tag{33}$$

where $f(x) = f(x + 2\pi/\Omega)$ and c is a constant. For simplicity we assume here that $f(x) := \alpha\cos(\Omega x)$, $\alpha \in \mathbb{R}$. The spatially periodic system (33) can be considered as the feedback interconnection of a spatially invariant system and a spatially periodic operator. Comparing (33) and (24) we have

$$\begin{aligned} &A^{\mathrm{o}} = -(\partial_x^2+1)^2 - c, \quad B^{\mathrm{o}} = C^{\mathrm{o}} = B = C = 1 \\ &D = 0, \quad F(x) = \alpha\cos(\Omega x). \end{aligned} \tag{34}$$

Fig. 5(a) illustrates the dependence of the $\mathcal{H}^2$ norm of this system on amplitude α and frequency Ω of the spatially periodic term for $c = 0.1$. The computations were performed by taking a large truncation of $\mathcal{A}_\theta$, $\mathcal{B}_\theta$, and $\mathcal{C}_\theta$ matrices and then using the Lyapunov method of Theorem 2.[3] For those pairs (Ω, α) that correspond to the points left blank in the plots of Fig. 5(a) the system is not stable and thus the $\mathcal{H}^2$ norm is not defined (or can be assumed to be infinite). For small values of α the $\mathcal{H}^2$ norm is close to the value of the $\mathcal{H}^2$ norm of the exponentially stable spatially invariant system corresponding to $\alpha = 0$. As α increases the $\mathcal{H}^2$ norm increases *for all* values of Ω. This increase is most significant at $\Omega \approx 2$. It is shown in [21], [23], [31] that this phenomenon can be interpreted as *parametric resonance* occurring between $f(x) = \alpha\cos(\Omega x)$, $\Omega = 2$, and a natural frequency at $k_x = 1$ of the spatially invariant system corresponding to $\alpha = 0$.

[3]To determine an appropriate truncation size N, we increase N from small values until the difference between the $\mathcal{H}^2$ norms of the two systems obtained from truncation sizes N and $2N$ becomes sufficiently small.

In many applications we are interested in *reducing* system norms via an appropriate choice of spatially periodic coefficients. We consider the system

$$\begin{aligned} \partial_t\psi &= -(\partial_x^2+1)^2\psi - c\psi + f\partial_x\psi + u \\ y &= \psi, \end{aligned} \tag{35}$$

with $f(x) = \alpha\cos(\Omega x)$ and $c = 0.1$. Again, comparing (35) and (24) we have the same system parameters as in (34) except that now $C^{\mathrm{o}} = \partial_x$. The system is exponentially stable for $\alpha = 0$. Fig. 5(b) shows that as α is increased from zero the $\mathcal{H}^2$ norm increases or decreases depending on the value of Ω. For $0 < \Omega \lesssim 1.2$ the $\mathcal{H}^2$ norm will *decrease* as the amplitude α of the periodic term is increased. This decrease is most significant at $\Omega \approx 0.5$. On the other hand, for $\Omega \gtrsim 1.2$ the $\mathcal{H}^2$ norm increases as α grows, this increase being most significant at $\Omega \approx 2$.

We now take $c = -0.1$ in (35), see Fig. 5(c). Notice that the system is *unstable* for $\alpha = 0$, and therefore its $\mathcal{H}^2$ norm is infinite. As α is increased from zero, the $\mathcal{H}^2$ norm remains infinite for $\Omega \gtrsim 1.2$. On the other hand, for $0 < \Omega \lesssim 1.2$ the $\mathcal{H}^2$ norm first becomes finite, and then decreases, as α grows in amplitude. Hence the periodic term effectively *stabilizes* the system. The best choice for Ω to most reduce the $\mathcal{H}^2$ norm is seen to be $\Omega \approx 0.5$.

We remark here that the above analysis can not be performed using conventional tools, e.g., Floquet analysis of periodic PDEs, as these methods do not easily lend themselves to the computation of system norms.

VIII. Conclusion

In this paper, we study the frequency response and input-output norm properties of spatially periodic LTI systems on a continuous unbounded spatial domain. We use frequency domain lifting to convert the linear spatially periodic system to a family of infinite-dimensional matrix problems. Such a representation lends itself readily to theoretical analysis and numerical computations. Using this framework we analyze the $\mathcal{H}^2$ norm of spatially periodic operators and systems, and present both deterministic and stochastic interpretations for it. We give examples of systems in which a spatially periodic feedback changes input-output norms, stabilizes an unstable system, or destabilizes a stable system.

In practice, the calculation of norms or analysis of spectral properties using the lifted representation involves taking large truncations of the lifted system operators $\mathcal{A}_\theta$, $\mathcal{B}_\theta$, $\mathcal{C}_\theta$ and $\mathcal{D}_\theta$. The size of these truncations depends on the system under consideration and can become increasingly large, which leads to expensive computations. This has led the authors to consider perturbation methods in the calculation of the $\mathcal{H}^2$ norm [21] and spectral analysis [23] for certain classes of spatially periodic systems. The perturbation analysis framework of [21] was recently employed to design an array of counter-rotating streamwise vortices to suppress turbulence in channel flows [32]. We have also used the lifted representation to explore other aspects

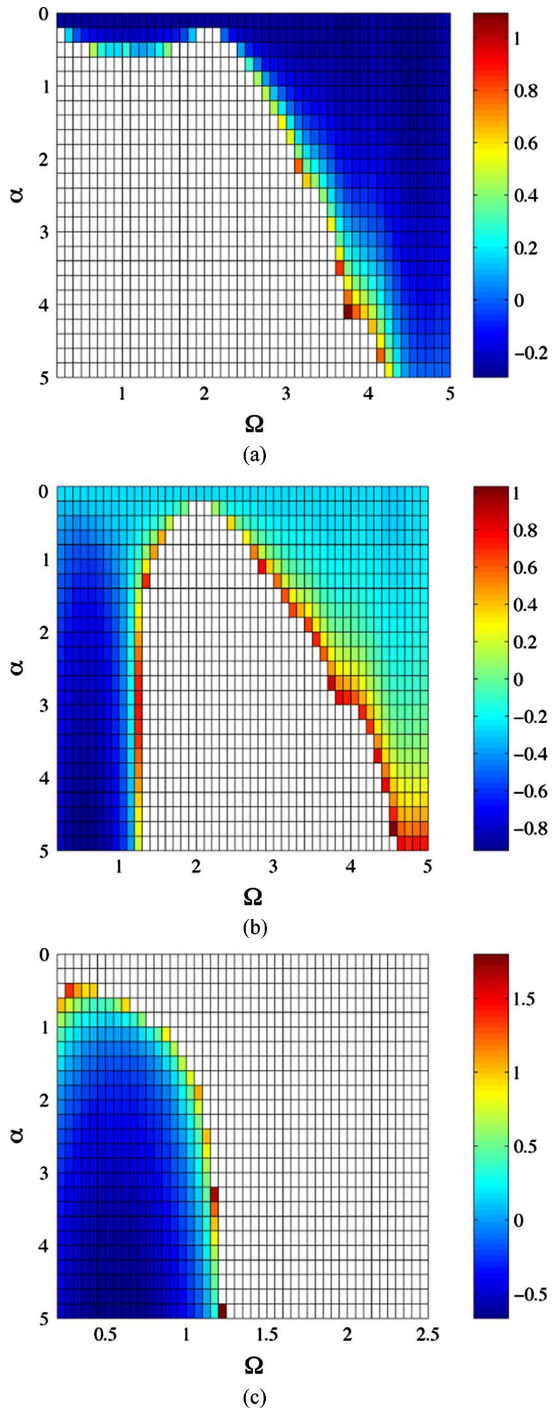


Fig. 5. (a) The log of the $\mathcal{H}^2$ norm of system (33) as a function of amplitude and frequency of the spatially periodic term for $c = 0.1$. (b) The log of the $\mathcal{H}^2$ norm of system (35) as a function of amplitude and frequency of the spatially periodic term for $c = 0.1$. (c) The log of the $\mathcal{H}^2$ norm of system (35) as a function of amplitude and frequency of the spatially periodic term for $c = -0.1$.

of spatially periodic systems such as parametric resonance [22], [23] and the Nyquist stability criterion [33].

While this paper focuses on spatially periodic systems in one spatial dimension for simplicity, [19] treats the problem in arbitrary spatial dimensions. It is known that for spatially distributed systems in spatial dimensions $d \geq 2$, exponential stability is not enough to guarantee finiteness of the $\mathcal{H}^2$ norm [34]. In [19] sufficient conditions are derived that guarantee a finite $\mathcal{H}^2$ norm for distributed systems in arbitrary spatial dimensions. For such systems one can also resort to the computation of two-point correlations as a substitute for the $\mathcal{H}^2$ norm. Finally, due to space limitations the $\mathcal{H}^\infty$ norm of spatially periodic systems was not addressed in this paper. The interested reader is referred to [19] for a detailed treatment.

APPENDIX

A. Spectral-Correlation Density Operators

Let u be a wide-sense stationary random field. We define its Fourier transform as $\hat{u}(k_x) = \int_{\mathbb{R}} e^{-jk_x x} u(x) dx$.[4] Let $R^u = \mathsf{E}\{u(x)u^*(\chi)\}$. Then it follows that:

$$\begin{aligned}\hat{S}^u(k_x,\kappa) &:= \mathsf{E}\{\hat{u}(k_x)\hat{u}^*(\kappa)\}\\ &= \int_{\mathbb{R}}\int_{\mathbb{R}} e^{-j(k_x x - \kappa\chi)} R^u(x,\chi) d\chi dx. \end{aligned} \quad \text{(A1)}$$

$\hat{S}^u$ is the Fourier transform of R^u and is called the *spectral-correlation density* of u. Since the random field u is wide-sense stationary we have $R^u(x,\chi) = R^u(x-\chi)$. Therefore from (A1) the spectral-correlation density of u assumes the form

$$\hat{S}^u(k_x,\kappa) = \hat{S}^u_0(k_x)\delta(k_x - \kappa)$$

where $\hat{S}^u_0(\cdot)$ is the spectral density of u. Heuristically, the above equation means that $\hat{u}$ is such an irregular function of frequency that no two samples $\hat{u}(k_x)$ and $\hat{u}(\kappa)$ of $\hat{u}$, $k_x \neq \kappa$, are correlated [15]. Notice that the spectral-correlation density $\hat{S}^u$ is a function of two frequency variables whereas the spectral density $\hat{S}^u_0$ is a function of only one frequency variable.

Next we consider a wide-sense *cyclostationary* random field u whose autocorrelation satisfies (8). The next theorem describes the structure of $\hat{S}^u$.

Theorem A1: Let u be a wide-sense cyclostationary random field with autocorrelation $R^u(x,\chi) = R^u(x + \mathrm{X}m, \chi + \mathrm{X}m)$, $m \in \mathbb{Z}$. Then u has the spectral-correlation density

$$\hat{S}^u(k_x,\kappa) = \sum_{l\in\mathbb{Z}} \hat{S}^u_l(k_x)\,\delta(k_x - \kappa - \Omega l) \quad \text{(A2)}$$

for some family of functions $\hat{S}^u_l$, $l \in \mathbb{Z}$.

Proof: See [19, Appendix to Chapter 2]. ■

Thus for a general cyclostationary signal the spectral-correlation density, as an operator in the Fourier domain, can be visualized as in Fig. 1(b).

The spectral-correlation density of the output of a periodic *operator* with a cyclostationary input is given by

$$\begin{aligned}\hat{S}^y(k_x,\kappa) &= \mathsf{E}\{\hat{y}(k_x)\ \hat{y}^*(\kappa)\}\\ &= \mathsf{E}\Big\{\int_{\mathbb{R}}\int_{\mathbb{R}} \hat{G}(k_x,\kappa_1)\hat{u}(\kappa_1)\hat{u}^*(\kappa_2)\hat{G}^*(\kappa_2,\kappa)\,d\kappa_1\,d\kappa_2\Big\}\\ &= \int_{\mathbb{R}}\int_{\mathbb{R}} \hat{G}(k_x,\kappa_1)\hat{S}^u(\kappa_1,\kappa_2)\hat{G}^*(\kappa_2,\kappa)\,d\kappa_1\,d\kappa_2\\ &= \big(\hat{G}\ \hat{S}^u\ \hat{G}^*\big)(k_x,\kappa)\end{aligned}$$

[4]Technically, the sample paths of wide-sense stationary signals are persistent and not finite-energy functions [15]. Hence their Fourier transforms fail to exist, i.e., $\int_{x\in\mathbb{R}} e^{-jk_x x}u(x)dx$ does not converge as a quadratic-mean integral. This problem can be circumvented by introducing the *integrated Fourier transform* [15], [35]. We will proceed formally and not pursue this direction here.

where $\hat{S}^u(\kappa_1,\kappa_2) := \mathsf{E}\{\hat{u}(\kappa_1)\hat{u}^*(\kappa_2)\}$, and $\hat{G}\hat{S}^u\hat{G}^*$ indicates the composition of kernels.

The spectral-correlation density of the output y of a linear spatially periodic LTI *system* with input u is given by

$$\begin{aligned}\hat{S}^y(t,\tau;k_x,\kappa) &= \mathsf{E}\{\hat{y}(t,k_x)\ \hat{y}^*(\tau,\kappa)\}\\ &= \mathsf{E}\Big\{\int \hat{G}(t,\tau_1;k_x,\kappa_1)\hat{u}(\tau_1,\kappa_1)\hat{u}^*(\tau_2,\kappa_2)\\ &\qquad \times \hat{G}^*(\tau_2,\tau;\kappa_2,\kappa)\ d\kappa_1\ d\kappa_2\ d\tau_1\ d\tau_2\Big\}\\ &= \int \hat{G}(t,\tau_1;k_x,\kappa_1)\hat{S}^u(\tau_1,\tau_2;\kappa_1,\kappa_2)\\ &\qquad \times \hat{G}^*(\tau_2,\tau;\kappa_2,\kappa)\ d\kappa_1\ d\kappa_2\ d\tau_1\ d\tau_2\\ &= \big(\hat{G}\ \hat{S}^u\ \hat{G}^*\big)(t,\tau;k_x,\kappa)\end{aligned}$$

where $\hat{S}^u(\tau_1,\tau_2;\kappa_1,\kappa_2) := \mathsf{E}\{\hat{u}(\tau_1,\kappa_1)\hat{u}^*(\tau_2,\kappa_2)\}$.

B. Proof of Theorem 1

Using the Plancharel theorem we have

$$\begin{aligned}\|G\|^2_{\mathcal{H}^2_{\rm sp}} &= \frac{1}{2\pi\mathrm{X}}\int_0^{\mathrm{X}}\int_{\mathbb{R}} \big(Gv^\chi\big)^*(x)\ \big(Gv^\chi\big)(x)\ dx\ d\chi\\ &= \frac{1}{2\pi\mathrm{X}}\int_0^{\mathrm{X}}\int\int\int_{\mathbb{R}} \big(\hat{v}^\chi(\kappa_1)\big)^*\ \hat{G}^*(\kappa_1,k_x)\\ &\qquad \times \hat{G}(k_x,\kappa_2)\ \hat{v}^\chi(\kappa_2)\ d\kappa_1\ d\kappa_2\ dk_x\ d\chi.\end{aligned}$$

Let $\hat{K}$ denote the composition $\hat{G}^*\hat{G}$, i.e., $\hat{K}(\kappa_1,\kappa_2) = \int_{\mathbb{R}}\hat{G}^*(\kappa_1,k_x)\hat{G}(k_x,\kappa_2)dk_x$. The exact expression for $\hat{K}$ is not required here; the important point is that $\hat{K}$ inherits the structure of $\hat{G}$ and $\hat{G}^*$, namely

$$\hat{K}(\kappa_1,\kappa_2) = \big(\hat{G}^*\hat{G}\big)(\kappa_1,\kappa_2) = \sum_{l\in\mathbb{Z}} \hat{K}_l(\kappa_1)\ \delta(\kappa_1-\kappa_2-\Omega l).$$

Using $\hat{v}^\chi(\kappa) = e^{-j\kappa\chi}$ and the structure of $\hat{K}$

$$\begin{aligned}\|G\|^2_{\mathcal{H}^2_{\rm sp}} &= \frac{1}{2\pi\mathrm{X}}\int_0^{\mathrm{X}}\int_{\mathbb{R}}\sum_{l\in\mathbb{Z}} e^{j\kappa_1\chi}\hat{K}_l(\kappa_1)\ e^{-j(\kappa_1-\Omega l)\chi}\ d\kappa_1\ d\chi\\ &= \frac{1}{2\pi}\int_{\mathbb{R}}\hat{K}_0(\kappa)\ d\kappa\end{aligned}$$

where we have used that $(1/2\pi\mathrm{X})\int_0^{\mathrm{X}} e^{j(\Omega l)x}dx$ is equal to one for $l=0$ and equal to zero for $l\neq 0$. It is not difficult to show that $\hat{K}_0(\kappa) = \sum_{l\in\mathbb{Z}}|\hat{G}_l(\kappa)|^2$, and thus

$$\begin{aligned}\|G\|^2_{\mathcal{H}^2_{\rm sp}} &= \frac{1}{2\pi}\sum_{l\in\mathbb{Z}}\int_{\mathbb{R}}|\hat{G}_l(\kappa)|^2\ d\kappa\\ &= \frac{1}{2\pi}\int_0^{\Omega}\mathrm{trace}\left[\mathcal{G}_\theta\mathcal{G}_\theta^*\right]\ d\theta.\end{aligned}$$

C. Proof of Theorem 2

We prove the theorem for $D=0$. Defining $u^\chi(t,x) := w(t)v^\chi(x) = \delta(t)\delta(x-\chi)$, [36], and proceeding formally

$$\begin{aligned}\|G\|^2_{\mathcal{H}^2} &= \frac{1}{\mathrm{X}}\int_0^{\mathrm{X}}\int_{\mathbb{R}}\int_0^\infty \mathrm{trace}\big[\big(Gu^\chi\big)\big(Gu^\chi\big)^*\big]\ dt\ dx\ d\chi\\ &= \frac{1}{2\pi\mathrm{X}}\int_0^{\mathrm{X}}\int_0^{\Omega}\int_0^\infty \mathrm{trace}\ \big[\big(\mathcal{G}_\theta(t)v_\theta^\chi\big)\big(\mathcal{G}_\theta(t)v_\theta^\chi\big)^*\big]\ dt\ d\theta\ d\chi\end{aligned}$$

where $\mathcal{G}_\theta(t) = \mathcal{C}_\theta e^{\mathcal{A}_\theta t}\mathcal{B}_\theta$. Using $(1/2\pi\mathrm{X})\int_0^{\mathrm{X}}\big(v_\theta^\chi\big)\big(v_\theta^\chi\big)^*\ d\chi = \mathcal{I}$ it follows that:

$$\begin{aligned}\|G\|^2_{\mathcal{H}^2} &= \frac{1}{2\pi}\int_0^{\Omega}\int_0^\infty \mathrm{trace}\left[\mathcal{G}_\theta(t)\mathcal{G}_\theta^*(t)\right]\ dt\ d\theta\\ &= \frac{1}{4\pi^2}\int_0^{\Omega}\int_{\mathbb{R}}\mathrm{trace}\left[\mathcal{G}_\theta(\omega)\mathcal{G}_\theta^*(\omega)\right]\ d\omega\ d\theta.\end{aligned}$$

Finally, it is a standard result of linear systems theory [24] that

$$\int_0^\infty \mathrm{trace}\left[\mathcal{C}_\theta e^{\mathcal{A}_\theta t}\mathcal{B}_\theta\mathcal{B}_\theta^* e^{\mathcal{A}_\theta^* t}\mathcal{C}_\theta^*\right]\ dt = \mathrm{trace}\left[\mathcal{C}_\theta\mathcal{P}_\theta\mathcal{C}_\theta^*\right]$$

where $\mathcal{P}_\theta$ is the solution of the (θ-parameterized) algebraic Lyapunov equation

$$\mathcal{A}_\theta\mathcal{P}_\theta + \mathcal{P}_\theta\mathcal{A}_\theta^* = -\mathcal{B}_\theta\mathcal{B}_\theta^*.$$

Therefore

$$\|G\|^2_{\mathcal{H}^2} = \frac{1}{2\pi}\int_0^{\Omega}\mathrm{trace}\left[\mathcal{C}_\theta\mathcal{P}_\theta\mathcal{C}_\theta^*\right]\ d\theta.$$

D. Proof of Theorem 3

Treating $R^y(\cdot,\cdot)$ as a kernel, applying the function $v^x(\chi) = \delta(\chi - x)$ to R^y from both sides, and using the Plancharel theorem and (30), we get

$$R^y(x,x) = \frac{1}{2\pi}\sum_{l\in\mathbb{R}}\int_{\mathbb{R}} e^{j(\Omega l)x}\ \hat{S}_l^y(k_x)\ dk_x.$$

Therefore

$$\begin{aligned}\frac{1}{\mathrm{X}}\int_0^{\mathrm{X}} R^y(x,x)\ dx &= \frac{1}{2\pi}\int_{\mathbb{R}}\sum_{l\in\mathbb{Z}}\left[\frac{1}{\mathrm{X}}\int_0^{\mathrm{X}} e^{j(\Omega l)x}dx\right]\\ &\qquad \times \hat{S}_l^y(k_x)\ dk_x\\ &= \frac{1}{2\pi}\int_{\mathbb{R}}\hat{S}_0^y(k_x)\ dk_x\\ &= \frac{1}{2\pi}\int_{\mathbb{R}}\sum_{l\in\mathbb{Z}}|\hat{G}_l(k_x)|^2\ \hat{S}_0^v(k_x)\ dk_x\end{aligned}$$

the last equality following from (31). If v is a white random field then $\hat{S}_0^v(k_x) = 1$ for all $k_x\in\mathbb{R}$ and we have

$$\frac{1}{\mathrm{X}}\int_0^{\mathrm{X}} R^y(x,x)\ dx = \frac{1}{2\pi}\int_{\mathbb{R}}\sum_{l\in\mathbb{Z}}|\hat{G}_l(k_x)|^2\ dk_x.$$

Finally, from the proof of Theorem 1 it follows that $\|G\|^2_{\mathcal{H}^2_{\rm sp}} = (1/\mathrm{X})\int_0^{\mathrm{X}} R^y(x,x)\ dx$.

E. Proof of Theorem 4

We use $\mathcal{E}_\theta := \mathcal{B}^{\rm o}_\theta \mathcal{F}\mathcal{C}^{\rm o}_\theta$, $\mathfrak{E}_\theta := \mathfrak{B}^{\rm o}_\theta \mathfrak{F}\mathfrak{C}^{\rm o}_\theta$, and $\mathcal{R}^{\rm o}_\theta := (j\omega\mathcal{I} - \mathcal{A}^{\rm o}_\theta)^{-1}$ to simplify notation. Our aim is to show

$$\int_0^\Omega \int_{\mathbb{R}} \|\mathcal{C}_\theta(j\omega\mathcal{I} - [\mathcal{A}^{\rm o}_\theta + \mathcal{E}_\theta])^{-1}\mathcal{B}_\theta - \mathfrak{C}_\theta(j\omega\mathcal{I} - [\mathcal{A}^{\rm o}_\theta + \mathfrak{E}_\theta])^{-1}\mathfrak{B}_\theta\|^2_{\rm HS}\ d\omega\ d\theta \to 0 \quad \text{(A3)}$$

as $N \to \infty$. From the identities

$$\begin{aligned}&\mathcal{C}_\theta(j\omega\mathcal{I} - [\mathcal{A}^{\rm o}_\theta + \mathcal{B}^{\rm o}_\theta\mathcal{F}\mathcal{C}^{\rm o}_\theta])^{-1}\mathcal{B}_\theta \quad \text{(A4)}\\ &= \mathcal{C}_\theta(j\omega\mathcal{I} - \mathcal{A}^{\rm o}_\theta)^{-1}\left(\mathcal{I} - \mathcal{B}^{\rm o}_\theta\mathcal{F}\mathcal{C}^{\rm o}_\theta(j\omega\mathcal{I} - \mathcal{A}^{\rm o}_\theta)^{-1}\right)^{-1}\mathcal{B}_\theta\\ &\mathfrak{C}_\theta(j\omega\mathcal{I} - [\mathcal{A}^{\rm o}_\theta + \mathfrak{B}^{\rm o}_\theta\mathfrak{F}\mathfrak{C}^{\rm o}_\theta])^{-1}\mathfrak{B}_\theta \quad \text{(A5)}\\ &= \mathfrak{C}_\theta(j\omega\mathcal{I} - \mathcal{A}^{\rm o}_\theta)^{-1}\left(\mathcal{I} - \mathfrak{B}^{\rm o}_\theta\mathfrak{F}\mathfrak{C}^{\rm o}_\theta(j\omega\mathcal{I} - \mathcal{A}^{\rm o}_\theta)^{-1}\right)^{-1}\mathfrak{B}_\theta\end{aligned}$$

we have

$$\begin{aligned}&\|\mathcal{C}_\theta(j\omega\mathcal{I} - [\mathcal{A}^{\rm o}_\theta + \mathcal{E}_\theta])^{-1}\mathcal{B}_\theta - \mathfrak{C}_\theta(j\omega\mathcal{I} - [\mathcal{A}^{\rm o}_\theta + \mathfrak{E}_\theta])^{-1}\mathfrak{B}_\theta\|_{\rm HS}\\ &\le \|\mathcal{C}_\theta\mathcal{R}^{\rm o}_\theta - \mathfrak{C}_\theta\mathcal{R}^{\rm o}_\theta\|_{\rm HS}\ \|(\mathcal{I} - \mathcal{E}_\theta\mathcal{R}^{\rm o}_\theta)^{-1}\mathcal{B}_\theta\|\\ &\quad + \|\mathfrak{C}_\theta(\mathcal{I} - \mathcal{R}^{\rm o}_\theta\mathfrak{E}_\theta)^{-1}\|\ \|\mathcal{R}^{\rm o}_\theta\mathcal{B}_\theta - \mathcal{R}^{\rm o}_\theta\mathfrak{B}_\theta\|_{\rm HS}\\ &\quad + \|\mathfrak{C}_\theta\mathcal{R}^{\rm o}_\theta\|\ \|(\mathcal{I} - \mathcal{E}_\theta\mathcal{R}^{\rm o}_\theta)^{-1}\|\\ &\quad \times \|\mathcal{E}_\theta\mathcal{R}^{\rm o}_\theta - \mathfrak{E}_\theta\mathcal{R}^{\rm o}_\theta\|_{\rm HS}\ \|(\mathcal{I} - \mathfrak{E}_\theta\mathcal{R}^{\rm o}_\theta)^{-1}\|\ \|\mathcal{B}_\theta\| \quad \text{(A6)}\end{aligned}$$

where we have used that $\|UTV\|_{\rm HS} \le \|U\|\|T\|_{\rm HS}\|V\|$ for $U, V \in \mathsf{B}(\ell^2)$ and $T \in \mathsf{B}_2(\ell^2)$ [16]. Thus

$$\begin{aligned}&\|\mathcal{C}_\theta(j\omega\mathcal{I} - [\mathcal{A}^{\rm o}_\theta + \mathcal{E}_\theta])^{-1}\mathcal{B}_\theta - \mathfrak{C}_\theta(j\omega\mathcal{I} - [\mathcal{A}^{\rm o}_\theta + \mathfrak{E}_\theta])^{-1}\mathfrak{B}_\theta\|^2_{\rm HS}\\ &\le \|\mathcal{C}_\theta\mathcal{R}^{\rm o}_\theta - \mathfrak{C}_\theta\mathcal{R}^{\rm o}_\theta\|^2_{\rm HS}\ \|(\mathcal{I} - \mathcal{E}_\theta\mathcal{R}^{\rm o}_\theta)^{-1}\mathcal{B}_\theta\|^2\\ &\quad + \text{square of other terms on right of (A6)}\\ &\quad + 2\|\mathcal{C}_\theta\mathcal{R}^{\rm o}_\theta - \mathfrak{C}_\theta\mathcal{R}^{\rm o}_\theta\|_{\rm HS}\ \|(\mathcal{I} - \mathcal{E}_\theta\mathcal{R}^{\rm o}_\theta)^{-1}\mathcal{B}_\theta\|\\ &\quad \times \|\mathfrak{C}_\theta(\mathcal{I} - \mathcal{R}^{\rm o}_\theta\mathfrak{E}_\theta)^{-1}\|\ \|\mathcal{R}^{\rm o}_\theta\mathcal{B}_\theta - \mathcal{R}^{\rm o}_\theta\mathfrak{B}_\theta\|_{\rm HS}\\ &\quad + \text{pairwise product of other terms on right of (A6)} \quad \text{(A7)}\end{aligned}$$

We present an argument regarding the first term on the right of (A7), with similar reasoning applying to the second and third terms. We prove that

$$\int_0^\Omega \int_{\mathbb{R}} \|(\mathcal{I} - \mathcal{E}_\theta\mathcal{R}^{\rm o}_\theta)^{-1}\mathcal{B}_\theta\|^2\ \|\mathcal{C}_\theta\mathcal{R}^{\rm o}_\theta - \mathfrak{C}_\theta\mathcal{R}^{\rm o}_\theta\|^2_{\rm HS}\ d\omega\ d\theta \to 0 \quad \text{(A8)}$$

as $N \to \infty$. Since G is exponentially stable, $\sup_{\omega,\theta}\ \|(\mathcal{I} - \mathcal{E}_\theta\mathcal{R}^{\rm o}_\theta)^{-1}\mathcal{B}_\theta\| = M$ for some $M > 0$. On the other hand

$$\begin{aligned}&\|\mathcal{C}_\theta\mathcal{R}^{\rm o}_\theta - \mathfrak{C}_\theta\mathcal{R}^{\rm o}_\theta\|^2_{\rm HS}\\ &= \|\mathcal{C}_\theta\mathcal{R}^{\rm o}_\theta \pm \mathcal{C}_\theta\Pi\mathcal{R}^{\rm o}_\theta - \Pi\mathcal{C}_\theta\Pi\mathcal{R}^{\rm o}_\theta\|^2_{\rm HS}\\ &\le \left(\|\mathcal{C}_\theta\|\ \|\mathcal{R}^{\rm o}_\theta - \Pi\mathcal{R}^{\rm o}_\theta\|_{\rm HS} + \|\mathcal{C}_\theta\Pi - \Pi\mathcal{C}_\theta\Pi\|\ \|\Pi\mathcal{R}^{\rm o}_\theta\|_{\rm HS}\right)^2\\ &= \|\mathcal{C}_\theta\|^2\ \|\mathcal{R}^{\rm o}_\theta - \Pi\mathcal{R}^{\rm o}_\theta\|^2_{\rm HS} + \|\mathcal{C}_\theta\Pi - \Pi\mathcal{C}_\theta\Pi\|^2\ \|\Pi\mathcal{R}^{\rm o}_\theta\|^2_{\rm HS}\\ &\quad + 2\ \|\mathcal{C}_\theta\|\ \|\mathcal{R}^{\rm o}_\theta - \Pi\mathcal{R}^{\rm o}_\theta\|_{\rm HS}\ \|\mathcal{C}_\theta\Pi - \Pi\mathcal{C}_\theta\Pi\|\ \|\Pi\mathcal{R}^{\rm o}_\theta\|_{\rm HS}.\end{aligned}$$

Therefore

$$\begin{aligned}&\int_0^\Omega \int_{\mathbb{R}} \|(\mathcal{I} - \mathcal{E}_\theta\mathcal{R}^{\rm o}_\theta)^{-1}\mathcal{B}_\theta\|^2\ \|\mathcal{C}_\theta\mathcal{R}^{\rm o}_\theta - \mathfrak{C}_\theta\mathcal{R}^{\rm o}_\theta\|^2_{\rm HS}\ d\omega\ d\theta\\ &\le M^2 \int_0^\Omega \int_{\mathbb{R}} \|\mathcal{C}_\theta\mathcal{R}^{\rm o}_\theta - \mathfrak{C}_\theta\mathcal{R}^{\rm o}_\theta\|^2_{\rm HS}\ d\omega\ d\theta\\ &\le M^2 \int_0^\Omega \int_{\mathbb{R}} \big(\|\mathcal{C}_\theta\|^2\ \|\mathcal{R}^{\rm o}_\theta - \Pi\mathcal{R}^{\rm o}_\theta\|^2_{\rm HS}\\ &\quad + \|\mathcal{C}_\theta\Pi - \Pi\mathcal{C}_\theta\Pi\|^2\ \|\Pi\mathcal{R}^{\rm o}_\theta\|^2_{\rm HS}\\ &\quad + 2\ \|\mathcal{C}_\theta\|\ \|\mathcal{R}^{\rm o}_\theta - \Pi\mathcal{R}^{\rm o}_\theta\|_{\rm HS}\\ &\quad \times \|\mathcal{C}_\theta\Pi - \Pi\mathcal{C}_\theta\Pi\|\ \|\Pi\mathcal{R}_\theta{}^{\rm o}\|_{\rm HS}\big)\ d\omega\ d\theta\\ &\le M^2\ \|C\|^2 \int_0^\Omega \int_{\mathbb{R}} \|\mathcal{R}^{\rm o}_\theta - \Pi\mathcal{R}^{\rm o}_\theta\|^2_{\rm HS}\ d\omega\ d\theta\\ &\quad + M^2\ \varepsilon^2 \int_0^\Omega \int_{\mathbb{R}} \|\Pi\mathcal{R}^{\rm o}_\theta\|^2_{\rm HS}\ d\omega\ d\theta\\ &\quad + 2M^2\|C\|\varepsilon \int_0^\Omega \int_{\mathbb{R}} \|\mathcal{R}^{\rm o}_\theta - \Pi\mathcal{R}^{\rm o}_\theta\|_{\rm HS}\\ &\quad \times \|\Pi\mathcal{R}^{\rm o}_\theta\|_{\rm HS}\ d\omega\ d\theta\\ &\le M^2\ \|C\|^2 \int_{|k_x|\ge K} \int_{\mathbb{R}} \left\|\left(j\omega I - \hat{A}^{\rm o}(k_x)\right)^{-1}\right\|^2_{\rm F}\ d\omega\ dk_x\\ &\quad + M^2\ \varepsilon^2 \int_{|k_x|\le K'} \int_{\mathbb{R}} \left\|\left(j\omega I - \hat{A}^{\rm o}(k_x)\right)^{-1}\right\|^2_{\rm F}\ d\omega\ dk_x \quad \text{(A9)}\end{aligned}$$

where K, K' are positive numbers that grow as the truncation size N is increased and $\varepsilon = \sup_\theta\ \|\mathcal{C}_\theta\Pi - \Pi\mathcal{C}_\theta\Pi\|$. Since by assumption $\int_{\mathbb{R}}\int_{\mathbb{R}} \|(j\omega I - \hat{A}^{\rm o}(k_x))^{-1}\|^2_{\rm F}\ d\omega\ dk_x$ is finite, the first integral in (A9) goes to zero as $N \to \infty$. Also, the boundedness of C implies that ε goes to zero as $N \to \infty$. Thus (A8) is proved. The same procedure can be applied to the integral of every term on the right of (A7) and therefore

$$\int_0^\Omega \int_{\mathbb{R}} \|\mathcal{C}_\theta(j\omega\mathcal{I} - [\mathcal{A}^{\rm o}_\theta + \mathcal{E}_\theta])^{-1}\mathcal{B}_\theta - \mathfrak{C}_\theta(j\omega\mathcal{I} - [\mathcal{A}^{\rm o}_\theta + \mathfrak{E}_\theta])^{-1}\mathfrak{B}_\theta\|^2_{\rm HS}\ d\omega\ d\theta \to 0$$

as $N \to \infty$, and (A3) is shown. In particular

$$\int_0^\Omega \int_{\mathbb{R}} \|\mathfrak{G}_\theta(\omega)\|^2_{\rm HS} d\omega d\theta \to \int_0^\Omega \int_{\mathbb{R}} \|\mathcal{G}_\theta(\omega)\|^2_{\rm HS} d\omega d\theta$$

which together with $\mathtt{G}_\theta(\omega) = \Pi\mathfrak{G}_\theta(\omega)\Pi\big|_{\Pi\ell^2}$ gives

$$\int_0^\Omega \int_{\mathbb{R}} \|\mathtt{G}_\theta(\omega)\|^2_{\rm F} d\omega d\theta \to \int_0^\Omega \int_{\mathbb{R}} \|\mathcal{G}_\theta(\omega)\|^2_{\rm HS} d\omega d\theta.$$

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Makan Fardad received the B.Sc. degree in electrical engineering from Sharif University of Technology, Tehran, Iran, the M.Sc. degree in electrical engineering from the Iran University of Science and Technology, Tehran, and the Ph.D. degree in mechanical engineering from the University of California, Santa Barbara, in 2006.

From 2006 to 2008 he was a Postdoctoral Associate with the Department of Electrical and Computer Engineering, University of Minnesota, Minneapolis. Currently he is an Assistant Professor of Electrical Engineering and Computer Science at Syracuse University, Syracuse, NY. His research interests are in the general area of the control of large-scale systems and systems governed by partial differential equations.

Mihailo R. Jovanović (S'00–M'05) received the Dipl.Ing. and M.S. degrees in mechanical engineering, from the University of Belgrade, Belgrade, Serbia, in 1995 and 1998, respectively, and the Ph.D. degree in mechanical engineering from the University of California, Santa Barbara, in 2004.

He was a Visiting Researcher with the Department of Mechanics, the Royal Institute of Technology, Stockholm, Sweden, from September to December 2004. He joined the University of Minnesota, Minneapolis, as an Assistant Professor of electrical and computer engineering in December 2004. His primary research interests are in modeling, analysis, and control of spatially distributed dynamical systems.

Dr. Jovanović received the CAREER Award from the National Science Foundation in 2007. He is a member of SIAM and APS and an Associate Editor of the IEEE Control Systems Society Conference Editorial Board.

Bassam Bamieh (F'08) received the Electrical Engineering and Physics degree from Valparaiso University, Valparaiso, IN, in 1983 and the M.Sc. and Ph.D. degrees from Rice University, Houston, TX, in 1986 and 1992, respectively.

From 1991 to 1998, he was with the Department of Electrical and Computer Engineering and the Coordinated Science Laboratory, University of Illinois at Urbana-Champaign. He is currently a Professor in the Mechanical Engineering Department, University of California at Santa Barbara. He is currently an Associate Editor of *Systems and Control Letters*. His current research interests are in distributed systems, shear flow turbulence modeling and control, quantum control, and thermo-acoustic energy conversion devices.

Dr. Bamieh received the AACC Hugo Schuck Best Paper Award, the IEEE CSS Axelby Outstanding Paper Award, and the NSF CAREER Award. He is a Control Systems Society Distinguished Lecturer.