So far we have studied stability of unforced systems of the form

\[ \dot{x} = f(x) \text{ or } \dot{x} = A\overrightarrow{x} \] (linear)

\[ Q: \text{is there any relation to stability of unforced system and stability w/ input?} \]

→ In linear case, stability of \( \dot{x} = A\overrightarrow{x} \)

(i.e., \( A: \) Hurwitz) implies that

"Bounded input means bounded output"

(Note: "Bounded input/Bounded output \( \Leftrightarrow A: \) Hurwitz"

\( \text{zero/pole cancellation} \Rightarrow \text{some modes are not controllable} \)

\[ \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u \]

\[ y = x_1 \]

\( \times_2 \) is NOT controllable OR observable \( \times \)
even though T.F. is stable
\[ \frac{Y(s)}{U(s)} = \frac{1}{s+2} \]

we do not have input to state stability

Q: Under what conditions does input to state stability ("bounded input/bounded output") guarantee all eigenvalues in LHP?

A: need to inject noise to check all transfer functions to observe zero/pole cancellation (see MIT Notes)

1. \[ Y_1(s) = \frac{H_1}{1 + H_1 H_2} \cdot U_1(s) \]
2. \[ E(s) = \frac{1}{1 + H_1 H_2} \cdot U_1(s) \quad \text{(No } H_1 \text{ in numerator)} \]
3. \[ E(s) = \frac{H_2}{1 + H_1 H_2} \cdot U_2(s) \]
4. \[ y_1(s) = \frac{-H_1H_2}{1+H_1H_2} u_1(s) \]

These are four T.F. to check out to guarantee stability in the modal sense.

So, why does stability of \[ \dot{x} = Ax \]
guarantee stability in input/output sense?

\[ \dot{x} = Ax + Bu \]

\[ x(t) = e^{At}x_0 + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau \]

**Fact:** If \( A \): Hurwitz (i.e., \( \text{Re}(\lambda(A)) < 0 \))

then

\[ \|e^{At}\| \leq K \cdot e^{-\alpha t} \]

\( (K, \alpha > 0) \)

\[ \|x(t)\| \leq \|e^{At}\|\|x_0\| + \int_0^t \|e^{A(t-\tau)}\| \|B\| \|u(\tau)\|d\tau \]

where \( \|\cdot\| \) is vector norm if \((\cdot)\) is a vector and an induced 2-norm if \((\cdot)\) is a matrix.
Recall the induced 2-norm of a matrix is the maximum singular value.

\[ \therefore \text{we can bound } \|e^{At}\| \]

\[ \therefore \text{if we assume bounded input} \]

\[ \text{then } \|x(t)\| \leq Ke^{-\lambda t} \|x_0\| + \frac{K}{\lambda} \|B\| \sup_{0 \leq t \leq T} \|u(t)\| \]

\[ \text{(effect of inputs)} \]

\[ \text{(effect of i.c.'s)} \]

A key here is that all eigenvalues of A lie in the left half of the complex plane.

\[ \text{Note: there is no robustness margin if you have any eigenvalues of the jIm-axis} \]

Consider the following example:

\[ \dot{x} = -x + xu \]

\[ \text{If } u = 0 \text{ then the system is stable} \]

\[ \dot{x} = -x \Rightarrow x(t) = e^{-t}x_0 \]
But if $\|u(t)\| = |u(t)| > 1$ for all $t$ we will have unbounded response.

E.g. if $u = 2$ then $\dot{x} = +x$, s.t.

$$x(t) = e^t x_0 \quad t \to \infty \quad \to +\infty$$

Def: Input to state stability: a system $\dot{x} = f(x, u)$ is input-to-state stable (ISS) if

$$\|\tilde{x}(t)\| \leq \beta(\tilde{x}_0, t) + \gamma \left( \sup_{\tau \in [0, t]} \|u(\tau)\| \right)$$

Where $\beta(\cdot)$ is a class-KL function and $\gamma(\cdot)$ is a class-K function.

Note: For linear systems,

1. $\beta(r, t) = K \cdot e^{-\alpha t} \cdot r \quad (K, \alpha > 0)$
2. $\gamma(s) = \frac{K}{\alpha} \|B\| \cdot s$
This is useful b/c if we know a system is ISS then we know that we have stability of the unforced system.

\[ \text{i.e. we already know we can bound } \|\dot{x}(t)\| \text{ by a class KL function} \]

\[ \text{i.e. if } \dot{x} = f(x, u) \text{ is ISS} \]
\[ \text{then } \dot{x} = f(x, 0) \text{ is G.A.S.} \]

Additionally if \( \dot{u}(t) \to 0 \) as \( t \to \infty \) then \( \dot{x}(t) \to 0 \) as \( t \to \infty \) as well.

How do we check if a system is ISS?

\[ \rightarrow \text{Lyapunov!} \]

\[ \text{These dissipation-like qualities are nothing more than Lyapunov comparison functions. That will help determine ISS.} \]
$\dot{x} = f(x, u)$ is ISS if there are four class $\mathcal{K}$ functions and a continuously differentiable $V(x)$ s.t.

1. $\alpha_1 \|x\| \leq V(x) \leq \alpha_2 \|x\|
2. $\dot{V} = \left[ \frac{\partial f}{\partial x} \right]^T f(x, u) \leq -\alpha_3 \|x\| + \alpha_4 \|u\|$

**Example:**

$x = -x^p + x^q \cdot u \quad \{p: \text{odd integer}\}$

This system will be ISS if $p > q$

(to be shown next lecture)

Assume $V(x) = \frac{1}{2} x^2$

then $\dot{V} = x \ddot{x} = -x^{p+1} + x^{2+1} u$

$p+1: \text{even}$

so this is $< 0$

potentially a source of trouble

→ use Young's Inequality!
Def: Young's Inequality

\[ a \cdot b \leq \frac{\alpha^r}{r} |a|^r + \frac{1}{S} \alpha^s |b|^s \]

\[ \{ \alpha > 0; \, r > 1; \, s > 1 \} \]

\[ (r-1)(s-1) = 1 \]

\[ (a - b)^2 \geq 0 \implies a^2 - 2ab + b^2 \geq 0 \]

\[ \therefore 2ab \leq a^2 + b^2 \]

and

\[ a \cdot b \leq \frac{a^2}{2} + \frac{b^2}{2} \]

\[ \rightarrow \text{However for } p, q \in \mathbb{R^+} \text{ where } \frac{1}{p} + \frac{1}{q} = 1 \]

\[ ab \leq \frac{a^p}{P} + \frac{b^p}{q} \]

Also called "Peter-Paul" Inequality

... "Rob Peter to pay Paul"