Today's Topics
- Feedback linearization
- Input-output linearization
- Zero dynamics
- Normal form

Recall the nonlinear dynamical system

\[ \dot{x} = f(x) + g(x) \cdot u \]
\[ y = h(x) \]

where \( \dot{x}(t) \in \mathbb{R}^n; \ y(t), u(t) \in \mathbb{R} \)

Relative degree is the number of differentiations needed for input \( u \) to arise in the equation.

- When relative degree is equal to number of states \( n \), we can completely linearize feedback.

- If relative degree \( < n \), we can linearize input to the output, but some part of the dynamics are not of this form (zero-dynamics).

**Def (Alt.):** A system of the form

\[
\begin{cases}
\dot{x} = f(x) + g(x) \cdot u \\
y = h(x)
\end{cases}
\]

has relative degree \( r \) if in a neighborhood around eq. pt. \( (x = 0) \).
(1) \( \log L f_i^{-1} h(x) = 0 \) for \( i = 1, \ldots, r-1 \)

(2) \( \log L f^{r-1} h(x) \neq 0 \)

Ex:
\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -x_1^3 + u
\end{align*}
\]

If \( y = x_1 \) \( \Rightarrow \dot{y} = x_2 \) \( \Rightarrow \ddot{y} = -x_1^3 + u \)

:: relative degree : \( r = 2 = n \)

This means that if we can measure the states we can completely remove nonlinearity with feedback \( \text{(i.e. } u = y^3 - k_1 \dot{y} - k_0 y) \)

s.t. \( \ddot{y} + k_1 \dot{y} + k_0 y = 0 \) \( (k_1, k_0 > 0) \)

Note: Modeling imperfections could be a huge source of trouble. The \( -x_1^3 \) term is largely stabilizing and cancelling it out may create problems with modeling imperfections

If \( u = -k_1 \dot{y} - k_0 y \) then
\[
\ddot{y} + k_1 \dot{y} + k_0 y + y^3 = 0
\]
The resulting input-output equation would not be linear, but we could still do trajectory-tracking

\[
\begin{cases}
    \dot{z}_1 = y \\
    \dot{z}_2 = g \quad \Rightarrow \\
    \ddot{z}_1 = z_2 \\
    \ddot{z}_2 = -k_0 z_1 - z_1^3 - k_1 z_2
\end{cases}
\]

Not linear but **GAS** for \( k_0 \geq 0 \) and \( k_1 > 0 \)  
(GES for \( k_0 > 0, \ k_1 > 0 \))

Let \( V(z) = \int_0^{z_1} g(\xi) \, d\xi + \frac{1}{2} z_2^2 \) s.t.

\[
V(z) = g(z_1) \dot{z}_1 + z_2 \ddot{z}_2
\]

\[
= g(z_1) z_2 - g(z_1) z_2 - z_2^2
\]

La Salle's invariance shows that if \( z_2 = 0 \) then

\[-z_1 (k_0 + z_1^2) = 0\]

only if \( z_1 = 0 \)

\[
\therefore \text{Globally asymptotically stable}
\]

Note GES may be better found w/ linearization
Q: What happens when relative degree $r < n$? (the number of states)

Consider the linear system:
\[
\begin{align*}
\dot{x} &= Ax + Bu \\
\dot{y} &= Cx
\end{align*}
\]

Note: $D = 0$ so there is no direct feedthrough term (i.e., we have roll off @ high frequency)

\[
\dot{y} = C\dot{x} = CAx + CBu
\]

\[\rightarrow\text{ for relative degree equal to 1}
\]
\[\text{then } CB \neq 0 \ (\exists^T b \neq 0 \text{ in SISO case})
\]

\[\rightarrow\text{ if not then differentiate again.}
\]

\[
\ddot{y} = CA\dot{x} = CA^2x + CABu
\]

\[\rightarrow\text{ relative degree } = 2
\]
\[\text{if } CAB \neq 0 \ (\exists^T A\bar{b} \neq 0 \text{ in SISO})
\]

\[\rightarrow\text{ if not then } \ldots \]

\[\text{Must keep going until } CAR^{-1}B \neq 0 \ (\exists^T AR^{-1}b \neq 0 \text{ in SISO})\]
Def: Markov Parameters

\[
D, CB, CAB, CA^2B, \ldots \text{ in } LTI
\]

In D.T. LTI,

\[
\begin{align*}
\dot{x}[k] &= A\dot{x}[k] + Bu[k] \\
\gamma[k] &= C\dot{x}[k] + Du[k]
\end{align*}
\]

Then these parameters characterize the impulse response matrix at different times k.

\[\Rightarrow\text{i.e., we introduce excitation in each individual input channel and record response. These responses stacked together will give impulse response matrix (in SISO, we only have one input channel, so we do not need this). This matrix is characterized by these Markov parameters}\]

Ex: \(D \in \mathbb{R}^{3 \times 2}, D \neq 0; \dot{x}[0] = 0\)

\[D = \begin{bmatrix}
d_{11} & d_{12} \\
d_{21} & d_{22} \\
d_{31} & d_{32}
\end{bmatrix}\]

\[\Rightarrow\text{How output will respond to input in first channel}\]

\[\Rightarrow\text{How output will respond to input in second channel}\]
in D.T. integrators work as delays

\[ \text{if relative degree is } r \text{ then we will not see the input until } r \text{ steps later} \]

Transfer function \( CRB + D \) is a \( z \)-transform of frequency response and therefore characterizes the same information.

- Let the system \( \begin{cases} \dot{x} = f(x) + g(x) \cdot u \\ y = h(x) \end{cases} \) have relative degree \( r < n \)

- Then \( y^{(r)} = L_f^r h(x) + L_g L_f^{r-1} h(x) \cdot u \)

\[ \text{[where } L_g L_f^{r-1} h(x) \neq 0 \text{ (in neighborhood } x = 0)\]}

\[ u = \frac{1}{L_g L_f^{r-1} h(x)} \left( -L_f^r h(x) + v \right) \]

This yields \( y^{(r)} = v \) (\( v \) is a new "input"").
\[ V = \sum_{i=0}^{r-1} k_i y(i) \quad \text{s.t.} \]
\[
y^{(r)} + k_{r-1} y^{(r-1)} + \ldots + k_0 y = 0
\]

Where we can choose \( k_i \) to provide stability of this subsystem.

\[ \text{But does this controller guarantee stability of the ORIGINAL SYSTEM?} \]

\[ \dot{x} = f(x) + g(x) \cdot u \]

NOT NECESSARILY! (Need additional conditions!)

This controller makes an \((n-r)\) dimensional manifold

\[ h(x) = Lf h(x) = \ldots = Lf^{r-1} h(x) = 0 \]

\[ y = y = \ldots = y^{(r-1)} = 0 \]
That is **invariant** (start there/stay there) and **attractive** (converge there as $t \to +\infty$).

→ This means we can force reference tracking for this system, but keeping system there does not always consider what the rest of the system is doing.

Zero-dynamics are the dynamics that emerge when we restrict output to this manifold.

→ These dynamics determine whether $x=0$ is stable.

**Example**:

Inverted pendulum on a cart.

4 states $\left\{ \begin{array}{l} y \\ \dot{y} \\ \dot{\theta} \\ \ddot{\theta} \end{array} \right\}$ w/ input $u$

MODEL:

\[
\begin{align*}
\dot{y} &= \frac{1}{M/m + \sin^2(\theta)} \left( \frac{1}{m} u + l \dot{\theta} \sin \theta - \frac{1}{2} g \sin \theta \right) \\
\dot{\theta} &= \frac{1}{l (M/m + \sin^2(\theta))} \left( -\frac{1}{M} u \cos \theta - \frac{l}{2} \dot{\theta}^2 \sin 2\theta + \frac{M+m}{m} g \sin \theta \right)
\end{align*}
\]
The relative degree is $2 < 4$ b/c $u$ appears in the equation for $\ddot{y}$.

If we keep output $y = 0$ we must cancel out all nonlinearities and add stabilizing terms.

Once we find this $u$ that keeps $y = 0$, plug that into $\ddot{y}$ to get the dynamics restricted to this domain.

Let $u = m \left( \frac{1}{2} g \sin 2\theta - l \dot{\theta} \sin \theta \right)$ then $y = 0$ and the remaining $\theta$ dynamics (zero dynamics) will be given by

$$\ddot{\theta} = \frac{g}{l} \sin \theta$$

This would be unstable for upright position even though $y$ is behaving appropriately, we CANNOT control $\theta \rightarrow$ zero-dynamics are uncontrolled.