Recall, for the system
\[
\begin{aligned}
\dot{x} &= f(x) + g(x) \xi \\
\dot{\xi} &= u
\end{aligned}
\]
\(\vec{x} \in \mathbb{R}^n, \xi, u \in \mathbb{R}\)

we can find \(u\) that stabilizes the system by finding \(\xi\) that stabilizes \(\dot{x}\) dynamics (upper system) for some Lyapunov function \(V\) and then augmenting the Lyapunov function (\(V_{aug}\)) to stabilize the whole system.

Example of "systems of triangular form"

\[
\begin{aligned}
\dot{x}_1 &= f_1(x_1) + g_1(x_1)x_2 \\
\dot{x}_2 &= f_2(x_1,x_2) + g_2(x_1,x_2)x_3 \\
\dot{x}_3 &= f_3(x_1,x_2,x_3) + g_3(x_1,x_2,x_3)u
\end{aligned}
\]

\(\text{... i.e., there is a \textit{cascade} of dynamical systems}\)
In linear case

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
\dot{x}_3
\end{bmatrix} =
\begin{bmatrix}
a_{11} & a_{12} & 0 \\
a_{21} & a_{22} & a_{23} \\
a_{31} & a_{32} & a_{33}
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix} +
\begin{bmatrix}
0 \\
0 \\
1
\end{bmatrix} u
\]

*Note:* Not to be confused with "triangular matrices"

We can also do **REFERENCE TRACKING**. Consider,

\[
\begin{align*}
\dot{x}_1 &= x_1^2 + x_2 \\
\dot{x}_2 &= u
\end{align*}
\]

where we want \( x_1(t) \to r(t) \) as \( t \to \infty \) (where \( r(t) \): given reference)

Introduce error variable \( z_1 := x_1 - r \) (\( \dot{z}_1 = \dot{x}_1 - \dot{r} \))
\[ z_1 = (z_1 + r)^2 + x_2 - \dot{r} \]

then propose \( V_1(z_1) = \frac{1}{2} z_1^2 \) s.t.

\[ \dot{V}_1 = z_1 \dot{z}_1 = z_1 \left[ (z_1 + r)^2 - \dot{r} + x_2 \right] \]

for \( \dot{V}_1(z_1) < 0 \ (\forall z_1 \neq 0) \) then

\[ \alpha_1(z_1) = - (z_1 + r)^2 + \dot{r} - K_1 z_1 \]

(\text{one option that would work})

Define \( z_2 := x_2 - \alpha_1(z_1) \) and \( V_{\text{aux}}(z_1, z_2) = V_1(z_1) + \frac{1}{2} z_2^2 \)

Then, \( \dot{V}_{\text{aux}} = \dot{V}_1 + z_2 \ddot{z}_2 = z_1 \left[ (z_1 - r)^2 - \dot{r} + z_2 + \alpha_1(z_1) \right] + z_2 \ddot{z}_2 \]

\[ = z_1 \left[ (z_1 - r)^2 - \dot{r} + \alpha_1(z_1) \right] + z_2 \left[ \ddot{z}_2 + z_1 \right] \]

\[ = z_1 \left[ (z_1 - r)^2 - \dot{r} + \alpha_1(z_1) \right] + (x_2 - \alpha_1(z_1)) \left[ \dot{x}_2 - \alpha_1(z_1) + x_1 - r \right] \]

\[ = z_1 \left[ (z_1 - r)^2 - \dot{r} + \alpha_1(z_1) \right] + (x_2 - \alpha_1(z_1)) \left[ u - \alpha_1(z_1) + x_1 - r \right] \]

\[ u = r + \dot{x}_1(x_1) - x_1 - K_2 (x_2 - \alpha_1(z_1)) \]

\[ = r - 2x_1 \dot{x}_1 + \ddot{r} - K_1 (x_1 - \dot{r}) - K_2 (x_2 + x_1^2 - \dot{r} + K_1 (x_1 - r)) \]

\[ = (1 + K_2) r + (K_1 + K_2) \dot{r} + \ddot{r} - (2x_1 + K_1 + K_2) (x_1^2 + x_2) - K_1 K_2 x_1 \]
Control Lyapunov Functions

Consider the system: $\dot{x} = f(x) + g(x) \cdot u$

then we propose the control law $u = \alpha(x)$

where $v = \left[ \frac{\partial V}{\partial x} \right]^T f(x) + \left[ \frac{\partial V}{\partial x} \right]^T g(x) \cdot \alpha(x) \leq 0$

... must be satisfied for stability!

Let $\left[ \frac{\partial V}{\partial x} \right]^T f(x) = \left[ \nabla V(x) \right]^T f(x) = L_f V(x)$

and $\left[ \frac{\partial V}{\partial x} \right]^T g(x) = \left[ \nabla V(x) \right]^T g(x) = L_g V(x)$

... These are called Lie Derivatives

→ can be thought of as an action of a gradient operating on a given vector (or vector field - e.g. $f(x)$ or $g(x)$)

[Note: LEFT-LIFTED ACTION]

$\dot{V} = L_f V(x) + L_g V(x) \cdot \alpha(x) \leq 0$
More generally, given positive definite, radially unbounded \( V(\vec{x}) \) find \( u \) s.t.

\[
\dot{V} = L_f V(\vec{x}) + L_g V(\vec{x}) \cdot u \leq 0
\]

Necessary condition for \( u \) to be able to meet this condition (i.e., \( V(\vec{x}) \) is a control Lyapunov function)

If for all \( \vec{x} \neq 0 \)

\[
L_g V(\vec{x}) = 0 \implies L_f V(\vec{x}) = 0
\]

**Example:**

\[
\begin{align*}
\dot{\vec{x}} &= \vec{x}^2 + u \\
V(\vec{x}) &= \frac{1}{2} \vec{x}^2
\end{align*}
\]

\[
\dot{V} = \vec{x} \dot{\vec{x}} = \vec{x}^3 + xu
\]

[trivial example but illustrates above point!]

Both equal to zero when \( \vec{x} = 0 \)

There are many ways to obtain stabilizing \( u = \alpha(\vec{x}) \)

E.g. Sontag's formula

\[
\alpha(\vec{x}) = \begin{cases} 
0 & \text{if } L_g V(\vec{x}) = 0 \\
- \frac{L_f V(\vec{x}) + \sqrt{(L_f V(\vec{x}))^2 + (L_g V(\vec{x}))^2}}{L_g V(\vec{x})} & ; L_g V(\vec{x}) \neq 0
\end{cases}
\]
Note: this is a continuous function of \( x \) if:

"Small control property" (See Khalil) holds

Feedback Linearization and Input/Output Linearization

(Relative degree and zero dynamics!)

Consider the system:

\[
\begin{align*}
\dot{x} &= f(x) + g(x) \cdot u \\
y &= h(x)
\end{align*}
\]

\[
\begin{align*}
x(t) &\in \mathbb{R}^n \\
u(t) &\in \mathbb{R} \\
y(t) &\in \mathbb{R}
\end{align*}
\]

Note: the above notions are essential for reference trajectory tracking

Relative degree lets us know generally how many times we have to differentiate output in order for the input to appear!

(i.e. How "far away" is the input from the output we care about?)
- If Relative degree = n:
  \[ \rightarrow \text{utilize feedback linearization} \]

- If Relative degree \( \leq n \):
  \[ \rightarrow \text{utilize input/output linearization} \]

**Zero dynamics:** part of the system that remains (i.e., is unobservable by the output)

**Def:** Relative degree: Number of times that we need to differentiate output \((y = h(x))\) in order for input \((u)\) to appear in the output equation.

**Ex:**
\[
\begin{align*}
  \dot{x}_1 &= x_2 \\
  \dot{x}_2 &= -x_1^3 + u \\
  y_1 &= x_1 \\
  y_2 &= x_2
\end{align*}
\]

\(y_1 = \dot{x}_1 = x_2\) \[\rightarrow \dot{y}_1 = \dot{x}_2 = -x_1^3 + u\] Relative Degree = 2

\(y_2 = \dot{x}_2\) Relative degree = 1
For example 1: Since relative degree $= n = 2$
we can make the system behave as a
linear system by completely cancelling out
nonlinearities by choice of $u$

\[ \dot{y}_1 = -x_1^3 + u \Rightarrow \text{Let } u = x_1^3 - k_1 y_1 - k_2 y_1 \]

such that we linearize by FIB

For example 2: Relative degree $= 2 < n$

There will be part of the dynamics which
is not observable. (Linearize by input/output)

This has implications to stability of reference tracking
as unobserved states may create instability

For the system:

\[
\begin{align*}
\dot{x} &= f(x) + g(x).u \\
y &= h(x)
\end{align*}
\]

then

\[
\begin{align*}
\dot{y} &= \left( \frac{\partial h}{\partial x} \right)^T \dot{x} = \left( \frac{\partial h}{\partial x} \right)^T f(x) + \left( \frac{\partial h}{\partial x} \right) g(x).u \\
&= Lf h(x) + Lg h(x).u
\end{align*}
\]

\[ \text{If } Lg h(x) \equiv 0 \text{ then we must continue differentiating!} \]
\[ \dot{y} = \frac{d}{dt} \left[ \begin{bmatrix} L_f \cdot h(\vec{x}) \end{bmatrix} \right] \]

\[ = \left[ \frac{2 L_f \cdot h(\vec{x})}{2 \vec{x}} \right]^T f(\vec{x}) + \left[ \frac{2 L_f \cdot h(\vec{x})}{2 \vec{x}} \right]^T g(\vec{x}) - u \]

\[ = L_f \cdot L_f \cdot h(\vec{x}) + L_g L_f h(\vec{x}) \cdot u \]

(Continued Next Lecture)