Recall the LTV system:
\[
\begin{align*}
\dot{x}_1 &= -a x_1 + \bar{w}^T(t) \bar{x}_2 \\
\dot{x}_2 &= -\bar{w}(t) x_1
\end{align*}
\]
where:
\[A(t) = \begin{bmatrix}
-a & \bar{w}^T(t) \\
\bar{w}(t) & 0
\end{bmatrix}
\]
\[\begin{bmatrix}
(A(t), C(t)) \\
(A(t) + L(t) C(t), C(t))
\end{bmatrix}
\]

Then,
\[P = \frac{1}{2} \mathbf{I} \implies \dot{V} = -\bar{x}^T C^T C \bar{x} \leq 0\]
where \[C = \begin{bmatrix}
\sqrt{a} & 0\end{bmatrix}\]

In order to show uniform asymptotic stability, we will need to show uniform observability of some system

\[\text{but state transition matrix is hard to compute!}\]
\[ A(t) + L(t) \mathbf{c}(t) = \begin{bmatrix} -a & \bar{w}^T(t) \\ -\bar{w}(t) & 0 \end{bmatrix} + \begin{bmatrix} l_1(t) \\ \bar{l}_2(t) \end{bmatrix} \begin{bmatrix} \sqrt{a} & 0 \end{bmatrix} \]

choose \( L(t) \) s.t.

\[ A(t) + L(t) \mathbf{c}(t) = \begin{bmatrix} 0 & \bar{w}^T(t) \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \sqrt{a} \\ \bar{l}_2(t) = \bar{w}(t) \end{bmatrix} \]

This matrix is much easier to work with!

\[ \Theta \Phi(t,t) = (A + LC) \Phi(t,t) = \begin{bmatrix} 0 & \bar{w}^T(t) \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix} \]

\[ = \begin{bmatrix} \bar{w}^T(t) \phi_{21} & \bar{w}^T(t) \phi_{22} \\ 0 & 0 \end{bmatrix} \]

\[ \begin{cases} \frac{2\phi_{11}}{2t} = \bar{w}^T(t) \phi_{21} \\ \frac{2\phi_{12}}{2t} = \bar{w}^T(t) \phi_{22} \\ \frac{2\phi_{21}}{2t} = 0 \\ \frac{2\phi_{22}}{2t} = 0 \end{cases} \]

where

\[ \begin{cases} \phi_{ii}(t_0,t_0) = I \\ \phi_{12}(t_0,t_0) = 0 \\ \phi_{21}(t_0,t_0) = 0 \end{cases} \]
1. $\phi_{22}(t, \tau) = I$

2. $\frac{2\phi_{12}}{2t} = \overrightarrow{w}_T(t) \cdot I$

   $\phi_{12}(t, t_0) - \phi_{12}(t_0, t_0) = \int_{t_0}^{t} \overrightarrow{w}_T(\tau) \, d\tau$

3. $\frac{2\phi_{21}}{2t} = 0 \implies \phi_{21}(t, t_0) = 0$

4. $\frac{2\phi_{11}}{2t} = \overrightarrow{w}_T(t) \cdot 0 = 0$

   $\phi_{11}(t, t_0) = \phi_{11}(t_0, t_0) = I_{1 \times 1} = I$

   $\phi_{11}(t, t_0) = \begin{bmatrix} 1 & \int_{t_0}^{t} \overrightarrow{w}_T(\tau) \, d\tau \\ 0 & I_{(n-1) \times (n-1)} \end{bmatrix}$

Does there exist $\delta, \alpha > 0$ s.t.

$$\int_{t_0}^{t_0 + \delta} \phi(t, \tau) C(\tau) C(\tau)^T \phi(t, \tau) \, d\tau \geq \alpha I$$

for all $t_0$?

Not necessarily the best way to solve this...
\[ \begin{align*}
\dot{x}_1 &= \bar{w}^T(t) \bar{x}_2 \\
\dot{x}_2 &= 0 \\
y &= \begin{bmatrix} \sqrt{a} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}
\end{align*} \]

\[ \begin{align*}
\dot{x}_1 &= -ax_1 + \bar{w}^T(t) \bar{x}_2 \\
\dot{x}_2 &= -\bar{w}^T \bar{x}_1 \\
y &= \begin{bmatrix} \sqrt{a} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}
\end{align*} \]

**Block Diagram**

\[ \begin{align*}
x_2 &= \text{const} \\
\bar{w}^T(t) &\rightarrow y_2 = u_1(t) \\
\int &\rightarrow x_1
\end{align*} \]

**Fact:** passing output through an integrator doesn't change uniform observability.

\[ \begin{align*}
\begin{align*}
\dot{x}_2 &= 0 \\
y_2 &= \bar{w}^T(t) \bar{x}_2
\end{align*}
\end{align*} \]
If $\exists \delta, \alpha > 0$ s.t.

$$\int_{t_0}^{t_0+\delta} \hat{w}(t) \hat{w}^T(t) dt \geq \alpha I$$

for all $t_0$, then G.U.A.S of original system.

**Model Reference Adaptive Control (MRAC)**

**Ex.** Consider the first order system

$$\dot{y} = ay + u$$  \hspace{1cm} (\alpha \text{ is unknown})

The control problem states: Design control input $u$ s.t.

$$\dot{y} = ay + u$$ behaves as a given reference model!

$$\dot{y}_m = -\alpha m y_m + r$$  \hspace{1cm} (\alpha_m > 0; r(t): \text{reference signal})

(m: model)

If $\alpha$ is known and $y(t)$ was measured then

set $u(t) = -(\alpha + \alpha_m) y(t) + r(t)$

s.t. $\dot{y} = ay - (\alpha + \alpha_m) y + r = -\alpha m y + r$
Define error as

\[ e(t) = y(t) - y_m(t) \implies \dot{e} = -a \cdot e \]

\[ \therefore e(t) = \exp(-a \cdot m \cdot t) \cdot e(0) \]

(i.e. the error will decay exponentially to zero)

What would we do if \( a = \text{constant but unknown} \)?

→ rather than using an estimator for \( a \), let's try to estimate the feedback gain so that the measured error still decays exponentially to zero!

→ (in previous example \( a: \text{known} \), \( K = a + a\cdot m \))

→ when \( a = \text{const. but unknown} \)

Let\[ u(t) = -\hat{K}(t) \cdot y(t) + r(t) \]

Let \[ \hat{K}(t) = K - \hat{K}(t) \] (FB gain estimation error)
where $K = \text{constant (unknown)}$

Note: we know from known a example, the feedback gain $K = a + am$ was constant. Therefore for $a = \text{constant BUT unknown}$ we would expect there to exist $K = \text{constant as well}$.

\[ y = a \cdot y - \hat{K}y + r = a \cdot y - (K - \hat{K})y + r \]
\[ = (a - K)y + \hat{K}y + r \]
\[ = -am \cdot y + \hat{K}(t)y + r \]

(Compare with $y_m = -amym + 0 \cdot ym + r$)

If $e(t) = y(t) - ym(t)$ and $\dot{e}(t) = y(t) - ym(t)$

then

\[ \dot{e}(t) = -am e(t) + \hat{K}(t) \cdot y(t) \]

(extra term that comes from estimation error for not knowing $K$)

\[ \text{How do we choose update law s.t. the } \hat{K} \cdot y \text{ term does not "hurt" our error?} \]
Only thing we can choose to design is $\hat{K}$
(y is measured output, e is error, and am is known)

must find equation for $\hat{K}$ that does not contain $K$ or $\hat{K}$

Start with Lyapunov function!

$$V(e, \hat{e}) = \frac{1}{2} e^2 + \frac{1}{2} \hat{e}^2$$

We want to drive measurement error (e) and FB gain error to zero!

$$\dot{V} = e \dot{e} + \hat{e} \dot{\hat{e}}$$

$$= -ame^2 + \hat{K}y e + \hat{K} \hat{e}$$

$$= -ame^2 + \hat{K} [\hat{e} + ey]$$

$$\leq 0 \quad \text{(Can this be negative?)}$$

Imagine that $\hat{K} = -ey$ then $\hat{K} [\hat{e} + ey] = 0$

Q: can we add a term to make this $\leq 0$?

$$\hat{K} = -ey - \alpha \hat{K}$$

A: NO! this depends on unknown $\hat{K}$
Adding this term leads to update law that is not implementable.

\[ \hat{k} = e \cdot y \text{ to eliminate this term!} \]

i.e. choose \( \hat{k} = -e \cdot y \) s.t. \( \dot{V} = -a_m e^2 \)