Today's Topics

- Convergence of Gradient update law for estimation of unknown parameters.

Linear parameterization in unknown params

\[ y(t) = \Psi^T(t) \cdot \hat{\Theta} \]

\( y \): Measured output (scalar)

\( \Psi^T(t) \): Regressor

\( \hat{\Theta} \): Constant but unknown params

\[ \hat{\Theta} := \tilde{\Theta}(t) + \hat{\Theta}(t) \]

We decompose \( \hat{\Theta} \)

Where \( \tilde{\Theta}(t) \) is an estimate and \( \hat{\Theta}(t) \) is the estimating error

\[ \dot{\Theta} = \dot{\hat{\Theta}} \]

Note:

\[ \frac{d\hat{\Theta}}{dt} = 0 \]

Then error between measured and estimated output is

\[ e(t) := y(t) - \hat{y}(t) = \Psi^T(t) \cdot \hat{\Theta}(t) \]

\[ \dot{\hat{\Theta}}(t) = -\Psi(t)\Psi^T(t) \cdot \hat{\Theta}(t) \]

[From gradient update of \( \hat{\Theta}(t) \)]
If $A(t) := -\Psi(t) \cdot \Psi(t)^T$ (i.e. $\dot{\tilde{\theta}} = A(t) \tilde{\theta}$) is uniformly asymptotically stable then

$$\lim_{t \to \infty} \tilde{\theta}(t) = 0$$

i.e.,

$$\lim_{t \to \infty} \dot{\tilde{\theta}}(t) = 0$$

Note: All signals are bounded.

$$\left\{ \begin{array}{l}
\dot{\tilde{\theta}} = -\Psi(t) \Psi^T(t) \tilde{\theta} \\
\tilde{\theta}(0) = \text{unknown initial condition (unknown)}
\end{array} \right.$$

$$\tilde{\theta}(0) = \tilde{\theta} - \dot{\theta}(0)$$

Under our control

Convenient for analysis, but not really for implementation

Implementation:

$$\dot{\theta} = -\dot{\theta} = -\Psi(t) \Psi^T(t) [\tilde{\theta} - \hat{\theta}]$$

$$= -\Psi(t) \Psi^T(t) \dot{\theta} + \Psi(t) \Psi^T(t) \hat{\theta}$$
\[
\dot{\hat{\theta}} = -\Phi(t)\Phi^T(t)\hat{\theta} + \Phi(t)y(t)
\]

where \( \hat{\theta}(0) \) is a given initial condition.

**Note:** \( y(t) \) will be measured scalar value. We can run this equation forward and we will have \( \dot{\hat{\theta}}(t) \) converge to \( \Theta \)

**ONLY** if \( A(t): -\Phi(t)\Phi^T(t) \) is stable (i.e., \( \hat{\theta} \) converges to zero!)

\[
\begin{array}{c}
\text{(Measured output)} \\
y(t) \\
\xrightarrow{\Phi(t)} \\
\Phi(t) \\
\text{(Observer Interpretation)} \\
\text{Consider the system} \\
\begin{cases}
\dot{x} = Ax + Bu \\
\hat{y} = Cx
\end{cases}
\end{array}
\]

\[
\begin{array}{c}
\text{Observer} \\
\begin{align*}
\dot{x} &= Ax + Bu + L(y - \hat{y}) \\
&= A\hat{x} + Bu + LC(x - \hat{x}) \\
\dot{x} &= (A - LC)\hat{x} + Bu + Ly
\end{align*}
\end{array}
\]
In our case...
\[ \hat{\theta} = 0 \]
\[ y(t) = \overline{\psi}^T(t) \hat{\theta} \]

\[ \Rightarrow \ A = B = 0 \]
\[ c(t) = \overline{\psi}^T(t) \]

Then
\[ \dot{\hat{\theta}} = (0 - L(t) \overline{\psi}^T(t)) \hat{\theta} + O \cdot \bar{u} + L(t) \cdot y(t) \]

If \[ L(t) = \bar{\psi}(t) \] then we regain parameter update law
\[ \dot{\hat{\theta}} = -\overline{\psi}(t) \overline{\psi}^T(t) \hat{\theta} + \overline{\psi}(t) y(t) \]

How would we study properties of \[ \dot{\hat{\theta}}(t) = -\overline{\psi}(t) \overline{\psi}^T(t) \hat{\theta} \]

If \[ \overline{\psi}(t) = \bar{\psi} = \text{constant} \]
\[ \dot{\hat{\theta}} = -\bar{\psi} \bar{\psi}^T \hat{\theta}; \hat{\theta} \in \mathbb{R}^p, \ p > 1 \]

LTI system where \[ A: -\bar{\psi} \bar{\psi}^T \in \mathbb{R}^{p \times p} \]

Is \[ A \] Hurwitz?
No!

There will only be one nonzero eigenvalue

\((\Psi \Psi^T \text{ will be rank one})\)

\[ -\Psi \Psi^T \bar{u}_i = \lambda_i \bar{u}_i \Rightarrow -\gamma_i \Psi = \lambda_i \bar{u}_i \]

\[ \therefore \begin{cases} \text{If } \bar{u}_i = \Psi \text{ then } \lambda_i = -||\Psi||^2 \\ \text{If } \bar{u}_i \in \{\text{span}(\Psi)\}^\perp \text{ then } \lambda_i = 0 \end{cases} \]

\[ \lambda_i(A) = \{-||\Psi||^2, 0, \ldots, 0\} \]

\[ p-1 \]

\[ \therefore \text{If } p = 2 \text{ then the system would be Marginal Stability} \]

\[ \therefore \text{If } p > 2 \text{ then the system would be Unstable} \]

\[ \therefore \Psi \Psi^T = A \text{ is Not Hurwitz and } \lim_{t \to \infty} \hat{\Theta}(t) \neq \hat{\Theta} \]
What if $\tilde{\Psi}(t)$ is time-varying?

As it turns out, YES!

USE LYAPUNOV BASED ANALYSIS!

For $\dot{\tilde{\Theta}} = -\tilde{\Psi}(t)\tilde{\Psi}^T(t)\tilde{\Theta}$

propose $V(\tilde{\Theta}) = \frac{1}{2}\tilde{\Theta}^T\tilde{\Theta} = \frac{1}{2}\tilde{\Theta}^T P \tilde{\Theta}$

(then $P = P(t) = \frac{1}{2} I = \text{constant}$)

then $\dot{V}(\tilde{\Theta}) = \frac{1}{2}\dot{\tilde{\Theta}}^T\dot{\tilde{\Theta}} + \frac{1}{2}\tilde{\Theta}^T\dot{\tilde{\Theta}}$

$= \tilde{\Theta}^T\dot{\tilde{\Theta}} = -\tilde{\Theta}^T\tilde{\Psi}(t)\tilde{\Psi}^T(t)\tilde{\Theta}$

$= -\tilde{\Theta}^T Q \tilde{\Theta}$

Where $Q(t) = \tilde{C}^T(t)\tilde{C}(t)$

where $\tilde{C}(t) = \tilde{\Psi}^T(t)$

Then $\dot{\tilde{\Theta}} = -\tilde{\Psi}(t)\tilde{\Psi}^T(t)\tilde{\Theta}$ is G.U.A.S. if $\exists \alpha > 0$

$s.t.$

$\int_{t_0}^{t_0+\delta} \tilde{\Theta}^T(t)\tilde{C}(t)\tilde{C}^T(t)\tilde{\Theta}(t)\Phi(t,t_0) \geq \alpha I$

for all $t_0 \geq 0$
i.e. our observability matrix has gained full rank on a given time interval for any initial conditions

How do we find $\Phi(t_{10})$?

... You don't!

Fact

$$(A(t), C(t))$$ is uniformly observable $\iff$ $$(A(t) - L(t)C(t), C(t))$$ is uniformly observable

i.e., Stability properties will not change with a choice of an observer

$$A(t) - L(t)C(t) = -\Psi(t)\Psi^T(t) - L(t)\Psi^T(t)$$

How do we choose $L(t)$ to make finding $\Phi(t_{10})$ trivial?

$$A(t) - L(t)C(t) = 0$$
\[ \text{if } L(t) = -\Psi(t) \text{ then } \]
\[ -\Psi(t)\Psi^T(t) + \Psi(t)\Psi^T(t) = 0 \]
\[ \{ \text{s.t. the dynamics are } \dot{\Theta} = 0 \} \]

\[ \Rightarrow \text{ The corresponding state transition matrix will be } \text{identity!} \quad (\Phi(t, t_0) = I) \]

\[ \therefore (A(t) - L(t)C(t), C(t)) \text{ is uniformly observable } \]
\[ (0, \Psi^T(t)) \text{ is uniformly observable } \] if
\[ \int_{t_0}^{t_0+\delta} \Psi(t)\Psi^T(t) \, dt \geq \alpha I \]

[Key here is that \( \Psi(t) \) determines convergence properties]

(i.e. \( \int_{t_0}^{t_0+\delta} \Psi(t)\Psi^T(t) \, dt \) must become full rank for any \( t_0 \) on a certain interval \([t_0, t_0+\delta]\).)
Ex1: \( \Psi(t) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \rightarrow \text{NO!} \quad \Psi(t) \Psi^T(t) = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \) is rank 1!

\[
\int_{t_0}^{t_0+\delta} \Psi(t) \Psi^T(t) \, dt = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \delta \neq I \cdot \alpha
\]

Ex2: \( \bar{\Psi}(t) = \begin{bmatrix} 1 \\ \sin(t) \end{bmatrix} \rightarrow \bar{\Psi}(t) \bar{\Psi}^T(t) = \begin{bmatrix} 1 & \sin(t) \\ \sin(t) & \sin^2(t) \end{bmatrix} \)

\[
\int_{t_0}^{t_0+\pi} \begin{bmatrix} 1 & \sin(t) \\ \sin(t) & \sin^2(t) \end{bmatrix} \, dt = \begin{bmatrix} 2\pi & 0 \\ 0 & \pi \end{bmatrix} \geq \pi I
\]

Where: \( \delta = 2\pi, \quad \alpha = \pi \) \( \text{YES} \)