Today's Topics

- Recap K/KL comparison functions
- Uniform Stability
- Stability using K/KL functions
- Exponential Stability

\[ \text{Class KL: } \beta(r,s) \]

Similarly thought of as a function of two positive scalar values; norm of I.C.'s and time \( t \)

\[ \text{Class K: } \alpha(r) \]

Can be thought of as a function of a positive scalar \( r \) and norm of initial conditions

\[ \text{Consider time varying systems} \text{ (and how they arise! ...)} \]

Ex:

\[
\begin{align*}
\dot{\bar{x}} & = f(\bar{z}) \\
\bar{z}(0) &= \bar{z}_0
\end{align*}
\]

Time-invariant nonlinear system

Let \( \bar{x}(t) \) be a solution to our system, and let \( \bar{z}(t) \) be fluctuations around \( \bar{z}(t) \)

\[
\bar{x}(t) := \bar{z}(t) = \bar{z}(t) - \bar{z}(t)
\]

\[ \text{Fluctuations} \quad \text{State of the system} \quad \text{given trajectory} \]

Then

\[
\dot{\bar{x}} = \dot{\bar{z}} - \dot{\bar{z}} = f(\bar{z}) - f(\bar{z})
\]

\[ = f'(\bar{x} + \bar{z}(t)) - f'(\bar{z}(t)) \]
Then if \( \bar{x} \) is a function of time (i.e. \( \bar{x}(t) \))
then \( \dot{\bar{x}} = f(\bar{x} + \bar{z}(t)) - f(\bar{z}(t)) \) is a
TIME VARYING system even though \( \dot{\bar{z}} = f(\bar{z}) \)
is time invariant!

**Ex**: optimization algorithms (Nesterov acceleration)

\[
\ddot{y} + \frac{t}{k} \dot{y} + \nabla f(\dot{y}) = 0 ; \quad \dot{y}(t) \in \mathbb{R}^n
\]

**Note**: Continuous dependence of parameters on time
is not met! \( \rightarrow \) need different tools to
establish existence and uniqueness

\( \rightarrow \) "Engineering band aid \( \rightarrow \frac{k}{t+\varepsilon} \)"

\[
\begin{cases}
\dot{x}_1 = y \\
\dot{x}_2 = \dot{y}
\end{cases}
\quad \rightarrow \quad \begin{cases}
\dot{x}_1 = x_2 \\
\dot{x}_2 = -\nabla f(x_1) - \frac{k}{t} x_2
\end{cases}
\]

**Ex**: Parameter estimation / identification
and Adaptive Control (More Later…)

Consider the nonlinear, time varying system

\( \dot{\bar{x}} = f(\bar{x}) \) where we assume \( \bar{x} = 0, \forall t \in [0, \infty) \)
(i.e., \( f(0, t) = 0 \) for all time) Then \( \bar{x} = 0 \) is **stable** (in Lyapunov sense) if

\[
\forall \varepsilon > 0, \ \exists \delta(\varepsilon, t_0) > 0, \text{ s.t. } \quad \|\bar{x}(t_0)\|_2 < \delta \implies \|\bar{x}(x_0, t_0)\|_2 < \varepsilon, \ \forall t \geq t_0
\]

\( \delta \) is a function of initial time!

**Note:** \( \bar{x}(x_0, t_0) := \phi(x_0, t, t_0) \) : [solution to our system starting at \( \bar{x}_0 \) at \( t = t_0 \)]

[In time invariant case, this notation simplifies to]

\[
\phi(\bar{x}_0, t-t_0)
\]

\( \delta = \delta(\varepsilon) \) [No dependence on \( t_0 \)]

\( \bar{x} = 0 \) is **uniformly stable**

The **SAME** formulation can be made for **Attractiveness** def. \( \bar{x} = 0 \) is **LAS** if it is **stable** (see above def) and:

\( \exists \delta_2 > 0 \) s.t. \( \|\bar{x}_0\|_2 < \delta_2 \implies \lim_{t \to \infty} \|\bar{x}(x_0, t_0, t)\|_2 = 0 \)
\[ \dot{x} = -\frac{1}{t+1} x = a(t) x \quad ; \quad x(t) \in \mathbb{R} \]

\[ \frac{dx}{x} = a(t) \, dt \quad \longrightarrow \quad \ln(x(t)) - \ln(x_0) = \int_{t_0}^{t} a(\tau) \, d\tau \]

\[ \therefore \quad x(t) = x_0 \, e^{\int_{t_0}^{t} a(\tau) \, d\tau} \]

\[ \int_{t_0}^{t} \left( -\frac{1}{t+1} \right) \, d\tau = \left[ -\ln|t+1| \right]_{t_0}^{t} = \ln|t_0+1| - \ln|t+1| \]

\[ = \ln \left| \frac{t_0+1}{t+1} \right| \]

\[ \therefore \quad x(t) = x_0 \cdot e^{\ln\left| \frac{t_0+1}{t+1} \right|} = x_0 \left( \frac{t_0+1}{t+1} \right) \]

Q: Is this uniformly stable? \[ \text{YES} \]

Rewrite the solution as:

\[ x(t) = x_0 \left( \frac{t_0+1}{(t-t_0) + t_0+1} \right) \]

\[ x(t) = x_0 \left( \frac{1}{1 + \frac{t-t_0}{t_0+1}} \right) \]

Note: \( t \geq t_0 \)
\[ t+1 \geq t_0+1 \implies \frac{t_{0+1}}{t+1} \leq 1 \]

\[ |x(t)| \leq |x_0| \]

\[ \rightarrow x = 0 \text{ is uniformly stable!} \]

\[ (\forall \varepsilon > 0, \text{ choose } \delta = \varepsilon) \]

\[ \rightarrow |x_0| < \delta = \varepsilon \]

\[ \text{then } |x(t)| \leq |x_0| < \varepsilon \]

Q: is this uniformly asymptotically stable? \[ \boxed{\text{NO}} \]

\[ \rightarrow \text{the solution is clearly asymptotically stable} \]

\[ \text{But do we have convergence independent of initial time?} \]

\[ |x(t)| \rightarrow |x_0| \text{ to } t \rightarrow t_0 \]

\[ x(t) = \frac{x_0}{1 + k(t_0)(t-t_0)} \]

(\text{where } k(t_0) \downarrow)

\[ \text{Note: } \lim_{t \to \infty} |x(t)| = 0 \text{ BUT the rate is influenced by choice of initial time} \]

\[ \therefore x = 0 \text{ is globally asymptotically stable, BUT NOT uniformly!} \]
i.e., the system is GAS but it is not uniformly GAS, because the convergence rate depends on to. (Solutions don't uniformly converge to $\bar{x}$ when to is varied...)

Note: Stability definitions for time-varying systems can be given in terms of comparison functions (much more convenient than $\epsilon$-$\delta$ definition!)

**def**

$\bar{x} = 0$ of $\dot{x} = f(x, t)$ is uniformly stable if there is a class K function $\alpha$ and a constant $c > 0$ s.t.

$$\|x(t)\| \leq \alpha(\|x(0)\|)$$

for all $t \geq t_0$ and all $x(t_0)$ with $\|x(t_0)\| < c$.

i.e., if we can bound the norm of $\dot{x}(t)$ by the increasing function $\alpha$ for all choices of $t_0$, then the system is uniformly stable (in Lyapunov sense).
**Def:** \( x=0 \) of \( \dot{x} = f(x,t) \) is uniformly asymptotically stable if there is a class KL function \( \beta \) and a constant \( c > 0 \) s.t.

\[
\|\bar{x}(t)\| \leq \beta (\|x(to)\|, t - to)
\]

for all \( t > to \) and all \( x(to) \) with \( \|x(to)\| < c \).

**Def:** If, in addition to uniformly asymptotically stable,

\[
\beta (r,s) = k \cdot r \cdot e^{-as} \quad (k, a > 0)
\]

then \( x=0 \) is uniformly exponentially stable

\[
\|\bar{x}(t)\| \leq k \cdot \|x(to)\| e^{-a(t-to)}
\]

→ DOES NOT IMPLY STRICTLY DECREASING (i.e. can have non-monotonic responses)

\[
K \|x(to)\| \quad \|\bar{x}(t)\| \quad t-to
\]

transient response (i.e. non-monotonically decreasing \( \|\bar{x}(t)\| \)