How do we construct Lyapunov functions?

**Example:**

\[ \dot{x} = -g(x) \quad \text{where} \quad x(t) \in \mathbb{R} \]

If suppose

\[ g(x) = x \quad \text{where} \quad x \cdot g(x) \geq 0 \quad \forall x \in [-a_1, a_2] \]

(i.e., \( g(x) \) belongs to first and third quadrant in some neighborhood of the origin)

\[ \Rightarrow \text{Note: in linear case } \quad g(x) = kx \quad \text{where } k > 0 \]

where \( x \cdot g(x) = kx^2 > 0 \quad \forall x \in \mathbb{R} \setminus \{0\} \)

Suppose we introduce \[ V(x) = \frac{1}{2} x^2 \]

\[ \begin{cases} 
V(0) = 0 \quad \text{and} \quad V(x) > 0 \quad \forall x \in \mathbb{R} \setminus \{0\} \\
V(x) = x \cdot \dot{x} = -x \cdot g(x) < 0 \quad \forall x \in [-a_1, a_2] 
\end{cases} \]

Therefore we can conclude **local asymptotic stability**
If \(-x \cdot g(x) < 0\) \(\forall x \in \mathbb{R} \setminus \{0\}\), then we can conclude **global asymptotic stability**.

Consider \(g(x) = \text{sat}(x)\).

\[\begin{array}{c}
\text{linear} \\
\text{quadratic}
\end{array}\]

Then \(x \cdot \text{sat}(x)\) behaves like

Huber function

\[\Rightarrow -x \cdot \text{sat}(x) < 0, \forall x \neq 0 \quad \text{and} \quad x = 0 \text{ is GAS}\]

**Aside:** Huber functions are particularly useful for minimizing the effect of outliers as opposed to least squares approximation of data.
As long as we have sector bounded nonlinearities (i.e., the function belongs to 1st and 3rd quadrant for some neighborhood) then we have local asymptotic stability for scalar systems!

For $x(t) \in \mathbb{R}^n$ for $n > 1$, more requirements must be met as well!

**Ex2:**

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -g(x_1) - bx_2
\end{align*}
\]

(for pendulum problems $g(x_1) = \sin(x_1)$)

- We have sector bounded nonlinearity for $g(x_1)$ for $x_1 \in [-\pi, \pi]$

**Let** $V(x) = \frac{1}{2} x_1^2 + \frac{1}{2} x_2^2$

1. $V(0) = 0$ and $V(x) > 0, \forall x \neq 0$

2. $\dot{V}(x) = x_1 \dot{x}_1 + x_2 \dot{x}_2 = x_1 x_2 - x_2 g(x_1) - bx_2^2 < 0$

How can we show negativity of cross terms?
\[ \dot{V}(\vec{x}) = x_1 x_2 - x_2 g(x_1) - bx_2^2 \leq \frac{1}{2} (x_1^2 + x_2^2) = \frac{1}{2} (x_2^2 + g(x_1)^2) - bx_2^2 \]

\[ \leq \frac{1}{2} x_1^2 + \frac{1}{2} g(x_1)^2 - bx_2^2 \]

[ Trouble is that we can't deal with \( x_1^2 + g(x_1)^2 \) ]

If we do not properly choose Lyapunov function candidate then this form of analysis may not reveal useful info.

Thus, this Lyapunov function candidate \( V(\vec{x}) = \frac{1}{2} x_1^2 + \frac{1}{2} x_2^2 \) is \( \not= \) a Lyapunov function of our system as we cannot conclude stability of \( \vec{x} = 0 \).

Consider instead:

\[ V(\vec{x}) = \int_0^{x_1} g(\xi) d\xi + \frac{1}{2} x_2^2 \]

Q: is \( V(\vec{x}) \) positive definite? A: Yes, but only locally!

Note \( \frac{d}{dt} \int_0^{x_1} g(\xi) d\xi = g(x_1) \cdot \dot{x}_1 \)

\[ \dot{V}(\vec{x}) = g(x_1) \cdot \dot{x}_1 + x_2 \dot{x}_2 \]

\[ = g(x_1) \cdot x_2 + x_2 [-g(x_1) - bx_2] \]

\[ = -bx_2^2 \leq 0 \quad \text{"around the origin"} \]
This is not negative definite because we do not have dependence on $x_1$.

\[ i.e. \text{ we can choose } \dot{x} = \begin{pmatrix} x_1 \\ 0 \end{pmatrix}; \quad x_1 \in \mathbb{R} \setminus \{0\} \]

\[ \text{s.t. } \dot{V}(x) = 0. \]

Because $\dot{V}(x)$ is negative semi-definite, we can only conclude stability in Lyapunov sense (i.e. no conclusions about asymptotic stability).

So then how do we further address stability when $\dot{V}(x)$ is negative semi-definite?

Invoke LaSalle's Invariance Principle!

**def:** LaSalle's Invariance Principle is a tool for assessing asymptotic stability properties of $x = 0$ for $\dot{x} = f(x)$ when $\dot{V}(x)$ is only negative semi-definite.

*(Goal: look for largest invariant sets ...)*

*Limitation is that this only holds for time invariant systems* if additional conditions hold!
Thm: Let \( \Omega_c := \{ x | V(x) \leq c^2 \} \) and \( \dot{V}(x) \leq 0, \forall x \in \Omega_c \).

Define \( S := \{ x \in \Omega_c | V(x) = 0 \} \) and let \( M \) be the largest invariant set in \( S \).

Then for every \( x(0) \in \Omega_c \) then

\[
\lim_{t \to \infty} x(t) \in M
\]

Corollary: If \( M := \{ 0 \} \), then we have asymptotic stability.

Recall for pendulum example

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -g(x_1) - bx_2
\end{align*}
\]

where \( V(x) = \int_0^x g(\xi) d\xi + \frac{1}{2} x_2^2 \). 

ie. \( V(x) \) is negative SEM definite

\[
\dot{V}(x) = -b x_2 x_1 - bx_2^2 \leq 0
\]

Then

\[
S := \{ (x_1, x_2) | x_2 = 0 \}
\]

ie. \( x_1 \) can be anything and \( S \) is the \( x_1 \)-axis

Q: What is the largest invariant set?
i.e. if we must follow the system's dynamics, then what is the largest set that satisfies these dynamics?

1. \[ x_2 = 0 \implies \dot{x}_2 = 0 \]

2. \[ \dot{x}_2 = 0 = -g(x_1) - b(0) = -g(x_1) \]

\[ \therefore \quad g(x_1) = 0 \text{ must be satisfied} \]

Locally, on the set of \( x_1 \in (-\pi, \pi) \) this is only satisfied for \( x_1 = 0 \).

Thus, \( M \) is the trivial set \( \bar{x} = 0 \).

\[ \therefore \quad \bar{x} = 0 \text{ is LAS} \]

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**Lasalle's Invariance Principle in the Context of Linear Systems.**

**Ex:**

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -x_1 - bx_2
\end{align*}
\]

where \( V(x) = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 \)

1. \( V(0) = 0 \) and \( V(x) > 0 \), \( \forall x \neq 0 \)

2. \( \dot{V}(x) = x_1 \dot{x}_1 + x_2 \dot{x}_2 = x_1 x_2 - x_1 x_2 - bx_2^2 = -bx_2^2 \leq 0 \)

(\( \dot{V} \) is negative semi-definite)
Here, Lyapunov + LaSalle shows GAS \((b \neq 0)\)

\[
\begin{align*}
\dot{x}_2 = 0 & \implies \dot{x}_2 = 0 \\
\dot{x}_2 = 0 &= -x_1 - b(0) = -x_1 \implies x_1 = 0
\end{align*}
\]

\[
\therefore \quad M := \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \implies \quad \begin{bmatrix} x = 0 \end{bmatrix} \text{ is GAS}
\]

\[\text{Note: in linear case } x = 0 \text{ is only eq. pt.}\]

\[
\therefore \quad \text{LAS } \implies \text{ GAS}
\]