Today's Topics
- Lyapunov functions
- La Salle's Invariance principle
- Stability via Linearization (proof)

(Generalizes Energy-Method)
(see pg. 63-64)

Recall,
\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{bmatrix} =
\begin{bmatrix}
x_2 \\
-sin x_1 - bx_2
\end{bmatrix}
\]

\[
E(x_1, x_2) = \int_0^x \sin(\xi) d\xi + \frac{1}{2} x_2^2
\]

\[
\frac{dE}{dt} = -0 \cdot x_1^2 - bx_2^2 \leq 0
\]

- No global stability because we have multiple eq. pts.

[Additionally, E is locally positive definite. (i.e., there is a neighborhood of \( x = 0 \) s.t. \( E(x) > 0 \), \( \forall x \in D \setminus \{0\} \)]

→ Main idea is that we introduce a energy-like function \( V \) of the state \( \dot{x} \), s.t.

\[ V : \mathbb{R}^n \rightarrow \mathbb{R}^+ \]

then study its derivative w.r.t. time along the trajectories \( \dot{x} = f(x) \)

\[
\frac{dV}{dt} = [\nabla V(x)]^T \cdot \dot{x}
\]

\[
= [\nabla V(x)]^T \cdot f(x) \leq 0
\]
Ex: Consider the 1D case:

\[ V(x) \]

\[ \exists \text{ level sets } V(x) = \text{const.} \]

\[ \rightarrow \text{"Ball" is representative of I.C.'s BUT it must behave according to } \dot{x} = f(x) \]

Ex: Consider \( \bar{x} \in \mathbb{R}^2 \)

\[ V(x_1, x_2) \geq 0 \]

\[ V = \text{const.} \]

\[ \text{for } \frac{dV}{dt} \leq 0 \text{ along trajectories we will either remain along the level set } (\frac{dV}{dt} = 0) \text{ or we will decay to lower level sets (or eq. pt.))} \]

Thm: (1) Let \( D \) be an open connected subset of \( \mathbb{R}^n \) that contains the eq. pt. \( \bar{x} = 0 \) for the system \( \dot{x} = f(x) \).

Then if there is a \textbf{continuously differentiable} function

\[ V: D \rightarrow \mathbb{R} \]

\[ a) \ V(0) = 0 \]

\[ b) \ V(\bar{x}) > 0 \ \forall \bar{x} \in D \setminus \{0\} \]

i.e. the function is \textbf{Locally positive definite}

and

\[ \frac{dV}{dt} = [\nabla V(\bar{x})]^T \cdot f(\bar{x}) \leq 0, \ \forall \bar{x} \in D \]
i.e., if $\frac{dV}{dt}$ is \textcolor{red}{negative semi-definite} (locally) 

[\textcolor{red}{Note: the "\leq" makes it semi-definite!}]

Then $\bar{x}=0$ is \textcolor{red}{stable in Lyapunov sense}

(2) In order to show $\bar{x}=0$ is \textcolor{red}{locally asymptotically stable}

then

\begin{itemize}
  \item[a)] $V(0)=0$
  \item[b)] $V(\bar{x})>0 \quad \forall \bar{x} \in D \setminus \{0\}$
  \item[c)] $\frac{dV}{dt} = [\nabla V(\bar{x})]^T f(\bar{x}) < 0, \quad \forall \bar{x} \in D \setminus \{0\}$
\end{itemize}

(\textcolor{red}{locally negative definite $V$})

(3) In order to show $\bar{x}=0$ is \textcolor{red}{globally asymptotically stable}

$D = \mathbb{R}^n$ s.t.

\begin{itemize}
  \item[a)] $V(0)=0$
  \item[b)] $V(\bar{x})>0, \quad \forall \bar{x} \in \mathbb{R}^n \setminus \{0\}$
  \item[c)] $\frac{dV}{dt} = [\nabla V(\bar{x})]^T f(\bar{x}) < 0, \quad \forall \bar{x} \in \mathbb{R}^n \setminus \{0\}$
  \item[d)] $\lim_{\|\bar{x}\| \to \infty} V(\bar{x}) = +\infty$ (i.e. radially unbounded)
\end{itemize}

\textcolor{red}{Global negative definite-ness of $\frac{dV}{dt}$!}

[\textcolor{red}{Note: we need to have existence and uniqueness of solutions (i.e., we need $f(\bar{x})$ to be locally Lipschitz)}]
Q: Why is radially unboundedness required for G.A.S.

Consider: \[ V(\bar{x}) = \frac{x_1^2}{1 + x_1^2} + x_2^2 \]

If \( \bar{x} = (x_1, s) \) then \( \|\bar{x}\| \to \infty \) as \( x_1 \to \infty \)

But \( V(\bar{x}) \) will approach \( \frac{26}{1 + x_1^2} \) Not radially unbounded

\[ \lim_{x_1 \to \infty} V(\bar{x}) = \lim_{x_1 \to \infty} \frac{x_1^2}{1 + x_1^2} + (s)^2 = 26 \neq +\infty \]

[possible trajectory that travels down level sets!]

\[ \text{level sets } V(\bar{x}) = \text{const.} \]

\[
\text{if } V(\bar{x}) < 0 \ (\forall \bar{x} \neq 0) \text{ But we do not have radial unboundedness, there is no guarantee trajectories will decay to } \bar{x} = 0 \text{ as } t \to \infty
\]

Proof (Sketch) of (1):

Consider the set \( \Omega_c := \{ \bar{x} | V(\bar{x}) = c \}^2 \)

where \( c = \text{constant} \) and \( \Omega_c \subset D \)
Note: because $V(x) \leq 0$, $\forall x \in D$, then each level set is positively invariant (i.e., if you start inside, you stay inside)

$\therefore \overline{x} = 0$ is stable (in Lyapunov sense)

**Proof (Sketch) of (2):**

$\dot{V}(x) < 0$, $\forall x \in D \setminus \partial \Omega$

\[
\begin{align*}
\text{Note: } V(x) & \text{ is a decreasing function of time } (\dot{V} < 0) \\
& \text{that is bounded below by } V(0) = 0 \\
\therefore \lim_{t \to \infty} V(x(t)) & \text{ exists}
\end{align*}
\]

Assume $\lim_{t \to \infty} V(x(t)) = C_2 \neq 0$

Let $D_1 := \{x | C_2 \leq V(x) \leq C_1\}$ s.t.

$$\max_{x \in D_1} \dot{V} = -\gamma \quad (\gamma > 0)$$

Then $\frac{dV}{dt} \leq -\gamma \Rightarrow V(x(t)) \leq V(x(0)) - \gamma \cdot t$

$\therefore \exists t$ s.t. $V(x(t)) < 0$

Thus $C_2 = 0$