Lyapunov Stability
(in short, stability wrt. I.C.'s ...)

Consider the time invariant system:
\[ \dot{x} = f(x) \]

There are **MANY** notions of stability

→ (in undergraduate control course, we may have had stability in the sense of "Bounded input, Bounded output")

But, we want to focus on stability w.r.t. I.C.'s (*No external input!*)

Consider the pendulum,

[Diagram of a pendulum]

How trajectories will evolve in the absence of external input will depend **SOLELY** ON I.C.'s

If \( x_0 = \bar{x} \) (i.e. if we do not start at an eq.p.t.) how are trajectories going to evolve as a function of time?

1) Assume existence/uniqueness
2) Assume Locally Lipschitz
3) Assume \( \bar{x} = 0 \)

**Note:** if \( f(\bar{x}) = 0 \) for \( \bar{x} \neq 0 \), then we introduce
change of variables \( \dot{\bar{x}}(t) = \dot{x}(t) - \bar{x} \) s.t.

\[
\dot{\bar{x}} = \dot{x} - \bar{x} = f(\bar{x}) - 0 = f(\bar{x} + \bar{x})
\]

\( \bar{x} = 0 \) is an eq. pt. of

\[
\dot{\bar{x}} = f(\bar{x} + \bar{x})
\]

Def: Let \( \bar{x} = 0 \) be an eq. pt. of \( \dot{x} = f(x) \), then:

1) \( \bar{x} \) is stable (in Lyapunov sense) if

\[
\forall \varepsilon > 0, \exists \delta_1(\varepsilon) > 0 \text{ s.t. } \\
\|\bar{x}_0\| < \delta_1 \implies \|\bar{x}(t, \bar{x}_0)\| < \varepsilon, \forall t \geq 0
\]

"Start close, stay close"

Note: this does not imply convergence!

Aside: If \( \bar{x} \neq 0 \) then the above rule states:

\[
\forall \varepsilon > 0, \exists \delta_1(\varepsilon) > 0 (\leq \varepsilon) \text{ s.t. } \\
\|\bar{x}_0 - \bar{x}\| < \delta_1 \implies \|\bar{x}(t, \bar{x}_0) - \bar{x}\| < \varepsilon, \forall t \geq 0
\]

→ can be avoided with coordinate transformation described above.
Note: Choice of finite dimensional norm changes "ball" of $\mathbb{R}^n$.

... this definition does not change w/ choice of $p$-norm.

2) $\bar{x}=0$ is **unstable** if (1) does not hold!

3) $\bar{x}=0$ is **locally asymptotically stable** if (1) holds AND

$$\exists \delta_2 > 0 \text{ s.t. } \| \bar{x}_0 \| < \delta_2 \Rightarrow \lim_{t \to \infty} \| \bar{x}(t, \bar{x}_0) \| = 0$$

i.e. LAS if Stable and Attractive

[Note: In nonlinear case (3) does not imply (1)!!
(i.e., attractiveness does not imply stability)]

$\Rightarrow$ **ex:** Homoclinic orbit

Homoclinic orbit: Non-periodic orbits that reach the same fixed pt as $t \to +\infty$
4) $\bar{x}=0$ is **globally asymptotically stable** if (1) and (3) hold

where $\delta_2 = \infty$.

i.e.,

\[
\|\bar{x}_0\| < \infty \implies \lim_{t \to \infty} \|\bar{x}(t, \bar{x}_0)\| = 0
\]

**Ex:** $\dot{x} = x(x-1) \rightarrow \bar{x} = 0$ and $\bar{x} = 1$

![Graph showing vector field and trajectories]

**..** $\bar{x} = 0$ is **locally asymptotically stable**

$\bar{x} = 1$ is **unstable**

"One way to check stability is to "plot" trajectories and see what is going on. This approach has limited utility because it requires solution to diff. eqn (which may be difficult to solve!)

... alternatively we can use Lyapunov-based analysis!

**Ex:** Consider the simple pendulum given by:

\[
\ddot{\theta} + b\dot{\theta} + a\cdot \sin \theta = 0
\]
Let \( \vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix} \), then

\[
\dot{\vec{x}} = \begin{bmatrix} \dot{\theta} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} x_2 \\ -a\sin(x_1) - b x_2 \end{bmatrix}
\]

\[\therefore \vec{x} = (0,0) \text{ AND } \vec{x} = (\pi,0) \]

("Down") \quad ("Up")

Q: What is the stability of the pendulum in "down" position?

Consider the energy of the system:

\[
E(x_1, x_2) = c_1 \int_0^{x_1} \sin(\xi) \, d\xi + \frac{c_2}{2} x_2^2
\]

Potential Energy \quad Kinetic Energy

→ What is the derivative of \( E \) along \( \vec{x} \)?

(i.e. is energy increasing or decreasing along trajectories?)

\[
\frac{dE}{dt} = (\nabla E)^T \frac{d\vec{x}}{dt} = \begin{bmatrix} \frac{\partial E}{\partial x_1} & \frac{\partial E}{\partial x_2} \end{bmatrix} \begin{bmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{bmatrix}
\]

\[
= \begin{bmatrix} c_1 \sin(x_1) \\ c_2 x_2 \end{bmatrix} \begin{bmatrix} x_2 \\ -a\sin(x_1) - b x_2 \end{bmatrix}
\]

\[
= c_1 x_2 \sin(x_1) - ac_2 x_2 \sin(x_1) - bc_2 x_2^2
\]

\[
= (c_1 - ac_2) x_2 \sin(x_1) - bc_2 x_2^2
\]
For \( c_2 = 1 \) and \( c_1 = a \), then

\[
\frac{dE}{dt} = -b x_z^2 \leq 0 \quad \text{(for } b \geq 0)\]

\[\text{NOTE: This only depends on } x_z \quad \text{(} x_1 \text{ can be ANYTHING!)}\]

\[\therefore \text{ Energy decreases along trajectories, and the eq. pt. } \bar{x} = (0,0) \text{ is stable (in Lyapunov sense!)}\]