Today’s Topics
- Continuous dependence on initial conditions and parameters
- Sensitivity eqns.
  (Lyapunov eqns.)

- Suppose we have $\dot{x} = f(x)$; $x(0) = x_0$
  Then if $f$ is locally Lipschitz cts.
  then we have existence and uniqueness of a solution on some interval $[0, tf)$
- Q: on some time interval, can we guarantee continuity w.r.t. I.C.’s?

Namely, given $\epsilon > 0$:

$\exists \delta > 0$ s.t. $\forall x^*_0 \in \{ x \in \mathbb{R}^n \mid ||x-y_0|| < \delta \} : = B_\delta$

1. $\Phi(x^*_0, t)$ is a unique solution
2. $||\Phi(x^*_0, t) - \Phi(y^*_0, t)|| < \epsilon$ for all $t \in [0, tf)$

**Ex:**

Even in linear systems, we cannot expect this to hold for $t \in [0, \infty)$

On the other hand, we can guarantee continuity w.r.t. I.C.’s on some finite interval.

Note: $\Phi(x^*_0, t)$: trajectory of the system $\dot{x} = f(x)$
with initial conditions $x^*_0 = x(0)$. 
What about continuity w.r.t. parameters?

→ Consider: \( \dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \bar{\mu}, t) \); \( (\bar{\mu}: \text{const. params}) \)

[Assume that \( \mathbf{f} \) is cts. w.r.t. \( \bar{\mu} \) AND around \( \bar{\mu} \).]

[\( \mathbf{f} \) is differentiable (\( \mathbf{f} \) is locally Lipschitz!)]

Rewrite the system as,

\[
\begin{cases}
\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \bar{\mu}, t) \\
\dot{\bar{\mu}} = 0 \quad (\bar{\mu}: \text{constant})
\end{cases}
\]

This new dynamical system \( \begin{bmatrix} \dot{\mathbf{x}} \\ \dot{\bar{\mu}} \end{bmatrix} = \mathbf{F}(\begin{bmatrix} \mathbf{x} \\ \bar{\mu} \end{bmatrix}, t) \)

can establish continuity w.r.t. parameter \( \bar{\mu} \) for \( \mathbf{f} \) by establishing continuity w.r.t. I.C.'s for \( \mathbf{F} \).

Thus, continuous dependence of trajectories on both I.C.'s and parameters on a finite time interval, by conversion

→ [If you have local Lipschitz continuity, then you have continuous dependence on I.C.'s for some time interval]

\[
\dot{\mathbf{z}} = \mathbf{F}(\mathbf{z}, t) \quad ; \quad \mathbf{z} = \begin{bmatrix} \mathbf{x} \\ \bar{\mu} \end{bmatrix} \quad \text{and} \quad \mathbf{F}(\mathbf{z}, t) = \begin{bmatrix} f(\mathbf{z}, t) \\ 0 \end{bmatrix}
\]
Existence and Uniqueness [+] IMPLIES Continuity w.r.t Initial Conditions

(Note: Locally Lipschitz $\Rightarrow$ Existence and Uniqueness)

Sensitivity w.r.t. Parameters.

Consider: \( \dot{x} = f(x, \bar{\mu}, t) \)

- where, \( f \): locally Lipschitz
  (we have continuity w.r.t IC and parameters)

- For the solution given by \( \bar{x} = \bar{\mu} \):

  \[ x(\bar{\mu}, t) \]

  How much would the solution change if we perturb \( \bar{\mu} \):

  \[ \bar{\mu} = \bar{\mu} + \hat{\mu} \] (\( \hat{\mu} \): perturbation)

- Approach: differentiate \( f \) w.r.t. parameters and study resulting differential equation

  \[ \ddot{x}(\bar{\mu}, t) = \ddot{x}(\bar{\mu}, t) + \frac{\partial \ddot{x}}{\partial \bar{\mu}} \Big|_{\bar{\mu}} (\bar{\mu} - \bar{\mu}) + \text{H.O.T.} \]

- Known trajectory \( S(t) \)
- Sensitivity function \( \ddot{\mu} \)
How does $S(t)$ change as a function of time?

For small changes in parameters

$$\bar{x}(\bar{u}, t) \approx \bar{x}(\bar{u}, t) + S'(t) \cdot (\bar{u} - \bar{u})$$

> Rewrite $\dot{x} = f(\bar{x}, \bar{u}, t)$ as

$$\bar{x}(\bar{u}, t) = \bar{x}(0) + \int_0^t f(\bar{x}(\bar{u}, \tau), \bar{u}, \tau) \, d\tau$$

... now take derivative w.r.t $\bar{u}$

**Then** w.r.t time!!

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**Note:** Let $\bar{x}_u(\bar{u}, t) := \frac{\partial \bar{x}(\bar{u}, t)}{\partial \bar{u}}$

1. Differentiating w.r.t $\bar{u}$.

$$\bar{x}_u(\bar{u}, t) = \frac{\partial \bar{x}(0)}{\partial \bar{u}} + \int_0^t \left( \frac{\partial f}{\partial \bar{x}} \cdot \frac{\partial \bar{x}}{\partial \bar{u}} + \frac{\partial f}{\partial \bar{u}} \right) \, d\tau$$

2. Now evaluate at $\bar{u}$:

$$f_u(\bar{x}(\bar{u}, t), \bar{u}, t) : function \ around \ known \ trajectory \ with \ parameter \ \bar{u}$$

$$\bar{x}_u(\bar{u}, t) : S(t)$$

? : Jacobian evaluated over trajectory $\bar{x}(\bar{u}, t)$. 

? :
\[ S(t) = \int_0^t \left[ A(\tau) S(\tau) + B(\tau) \right] d\tau \]

where \( A(\tau) = \frac{\partial f}{\partial x}(x(\mu, \tau), \bar{\mu}, \tau) \) and \( B(\tau) = \frac{\partial f}{\partial \mu}(x(\mu, \tau), \bar{\mu}, \tau) \)

\[ \frac{d S(t)}{dt} = A(t) S(t) + B(t) \]

**Example:** Fold Bifurcation

\[ x = x^2 + \mu = f(x, \mu) \]

Then \( \frac{\partial f}{\partial x} = 2x \) and \( \frac{\partial f}{\partial \mu} = 1 \)

\[ \dot{S}(t) = 2x(\bar{\mu}, t) \cdot S(t) + 1 \]

Again, unlikely to have analytical solution, but in general we can create \( S(t) \) in parallel:

1. \( \left\{ \begin{array}{l} x = x^2 + \bar{\mu} \\ \dot{S} = 2x \cdot S + 1 \end{array} \right. \)

2. not coupled to \( S \)!

Solve \( x(\bar{\mu}, t) \) with some ODE solver

Then feed into here and solve!
Ex2: (From Khalil)

\[
\begin{cases}
\dot{x}_1 = x_2 = f_1(x, \mu) \\
\dot{x}_2 = -c \cdot \sin(x_1) - [a + b \cos(x_1)] x_2 = f_2(x, \mu)
\end{cases}
\]

thus \[\mu := \begin{bmatrix} a \\ b \\ c \end{bmatrix} ; \quad \bar{x}(t) \in \mathbb{R}^2\]

→ Study sensitivity when \[\mu = [1, 0, 1]^T\].

\[
\frac{\partial f}{\partial \bar{x}} = \begin{bmatrix}
\frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\
\frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2}
\end{bmatrix} = \begin{bmatrix}
0 & 1 \\
-c \cdot \cos(x_1) + b \cdot \sin(x_1) x_2 & -[a + b \cdot \cos(x_1)]
\end{bmatrix}
\]

\[
\frac{\partial f}{\partial \mu} = \begin{bmatrix}
\frac{\partial f_1}{\partial a} & \frac{\partial f_1}{\partial b} & \frac{\partial f_1}{\partial c} \\
\frac{\partial f_2}{\partial a} & \frac{\partial f_2}{\partial b} & \frac{\partial f_2}{\partial c}
\end{bmatrix} = \begin{bmatrix}
0 & 0 & 0 \\
-x_2 & -x_2 \cos(x_1) & -\sin(x_1)
\end{bmatrix}
\]