Today's Topics
- Existence & Uniqueness of solutions
  - Continuous dependence on initial conditions
  - Parameters

[Material can be found in Khalil Chapter 3] (See textbook for more rigorous def.)

Existence AND Uniqueness of solutions to:
\[
\begin{align*}
\dot{x} &= f(x) \\
\dot{x} &= f(x, t) \quad \text{(time varying)}
\end{align*}
\]

→ In Linear Case: \[ \dot{x} = Ax \] or \[ \dot{x} = A(t)x \] respectively

→ In the static case: Given \( A \in \mathbb{R}^{m \times n} \) and \( \bar{b} \in \mathbb{R}^m \)

\[ A\bar{x} = \bar{b} \]

This equation can be solved only if \( \bar{b} \in \text{col}(A) \) (i.e. if \( \bar{b} \) is in Range(A)

where \( \text{Range}(A) = \text{span}\{ \bar{a}_1, \ldots, \bar{a}_n \} \)

\[ A\bar{x} = [\bar{a}_1 \ldots \bar{a}_n] \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \sum_i x_i \bar{a}_i \]

→ i.e. there is at least one solution (existence)

→ if \( \text{Null}(A) = \{ 0 \} \) then solution is unique

\[ \text{Null}(A^2) = \{ 0 \} \quad (A\bar{x} = 0 \bar{x}) \]
How do we extend this to LTI system:

Consider: \( \dot{x} = Ax \)

Then the solution is given by \( \bar{x}(t) = e^{At}\bar{x}(0) \)

And is unique!

For time-varying case, \( \dot{x} = A(t)x \), we need to "restrict classes" of \( A(t) \) to guarantee existence and uniqueness.

\[ \Rightarrow \text{Consider } A(t) = [a_{ij}(t)] \text{ where elements are functions of time.} \]

\[ \Rightarrow a_{ij}(t) \text{ MUST BE PIECEWISE CONTINUOUS!} \]

.: Regarding time-dependence, if \( f(\bar{x}, t) \) is a piecewise-continuous of time, then "we'll be fine".

\[ \left[ \begin{array}{c} \text{we will only consider piecewise-cts functions} \\ \text{of time when assessing existence/uniqueness} \end{array} \right] \]

(i.e., this is an extension of the restriction to linear case where \( a_{ij}(t) \) are piecewise-cts functions of time)

Q: What restrictions w.r.t. \( \bar{x} \) shall we introduce? And is continuity enough?
Consider: \( \dot{x} = x^{1/3} \); \( f(x) = x^{1/3} \): cts-function!

Yes! Solution exists for \( \bar{x}(0) = 0 \) \( \Rightarrow \bar{x}(t) = 0 \)

**But** there is another solution where
\[
\hat{x}(0) = 0 \Rightarrow \hat{x}(t) = \left( \frac{2t}{3} \right)^{3/2}
\]
\[
(\text{i.e., } \hat{x}(t) = \frac{3}{2} \left( \frac{2t}{3} \right)^{1/2} \cdot \frac{12}{3} = \left( \frac{2t}{3} \right)^{1/2} = \left( \frac{2t}{3} \right)^{3/2} \cdot \frac{1}{3} = x^{1/3})
\]

"Fact!" If \( f(x,t) \) is a continuous function of \( x \) and piecewise function of time, then WE have existence of solutions

\[
\because \text{We need additional restrictions on } f(x,t) \text{ in order to show uniqueness!}
\]

Note: \( f(x) = x^{1/3} \) is **not** continuously differentiable

\[
\text{Infinite slope at the origin will create problems!}
\]
Def: Lipschitz continuous functions satisfy
\[ \| f(x) - f(y) \| \leq L \cdot \| x - y \| \]

→ function can be **locally Lipschitz** if there exists a positive constant \( L \) that satisfies this condition in a specific neighborhood.

→ additionally, this can hold everywhere and the function would be **globally Lipschitz**.

**Local Lipschitz** if \( \exists L > 0 \) s.t. \( \| f(x) - f(y) \| \leq L \| x - y \| \), \( \forall x, y \in \mathbb{R}^n \) where \( \| x \|, \| y \| \leq r \)

**Global Lipschitz** if \( \exists L > 0 \) s.t. \( \| f(x) - f(y) \| \leq L \| x - y \| \), \( \forall x, y \in \mathbb{R}^n \)

Ex: \( f(x) = x^2 \)

Then \( | f(x) - f(y) | = | x^2 - y^2 | = | (x+y)(x-y) | \leq |x+y| \cdot |x-y| = L(x,y) \cdot |x-y| \)

- There is not value of \( L \in \mathbb{R} \) s.t. this function is Lipschitz **for all** \( x, y \).
- But we could find \( L \) (as a function of the operating point: \( \bar{z} \)) s.t. Lipschitz holds locally.
i.e. we can always choose $L(x,y)$ (in this example) to be some number that bounds $|x+y|$ where $x, y \in B_r$ (some neighborhood).

Ex: $\dot{x} = -x^3$

$$|f(x) - f(y)| = |y^3 - x^3|$$

$$= |(y^2 + xy + x^2)(x-y)|$$

$$\leq |y^2 + xy + x^2| \cdot |x-y|$$

Again this can be **locally Lipschitz** (But not **globally**)

Yet, we have **existence and uniqueness** on some time interval on $[0, \infty)$

"**Fact 2**": If $f(\bar{x}, t)$ is **locally Lipschitz continuous** w.r.t. $\bar{x}$, then we have **existence** and **uniqueness** on time interval $[0, tf)$

"**Fact 3**": If $f(\bar{x}, t)$ is **globally Lipschitz, continuous** w.r.t. $\bar{x}$, then we have **existence** and **uniqueness** on time interval $[0, \infty)$

\[\textbf{Issue:} \text{ this condition is difficult to satisfy (see example above)}\]
Fact 4: ANY continuously differentiable function $f(x)$ is locally Lipschitz continuous.

[i.e. if a function is differentiable and its derivative is continuous]

Example:

- $x = x^2 \implies \frac{df}{dx} = 2x \implies$ Locally Lipschitz!

- $x = x^3 \implies \frac{df}{dx} = 3x^2 \implies$ Locally Lipschitz!

- $x = x^{1/3} \implies \frac{df}{dx} = \frac{1}{3x^{2/3}} \implies$ Not continuous! Blows up at $x = 0$. Not locally Lipschitz.

Note: these conditions are sufficient but not necessary!

- There are functions that are globally Lipschitz continuous, but not differentiable.

Example:

$f(x) = \text{sat}(x)$

The derivative is undefined at $x = \pm 1$. 

Diagram showing the function and its derivative.
But we can choose $L \geq 1$ s.t.

\[ \text{... i.e. } |L \cdot x| \geq |\text{sat}(x)| \]

$C^0$:
- Continuous
- $L$-Lipschitz
- $C^1$-Differentiable