

A Robust Optimization Approach for the Capacitated Vehicle Routing Problem with Demand Uncertainty*

Abstract

In this paper we introduce a robust optimization approach to solve the Vehicle Routing Problem (VRP) with demand uncertainty. This approach yields routes that minimize transportation costs while satisfying all demands in a given bounded uncertainty set. We show that for the Miller-Tucker-Zemlin formulation of the VRP and specific uncertainty sets, solving for the robust solution is no more difficult than solving a single deterministic VRP. We present computational results that investigate the trade-offs of a robust solution for the Augerat et al. suite of capacitated VRP problems and for families of clustered instances. Our computational results show that the robust solution can protect from unmet demand while incurring a small additional cost over deterministic optimal routes. This is most profound for clustered instances under moderate uncertainty, where remaining vehicle capacity is used to protect against variations within each cluster at a small additional cost. We observe that the robust solution amounts to a clever management of the remaining vehicle capacity.

Keywords: Robust optimization; Vehicle routing; Demand uncertainty

*Research supported by NSF under grant CMS-0409887

1 Introduction

Many industrial applications deal with the problem of routing a fleet of vehicles from a depot to service a set of customers that are geographically dispersed. This type of problem is faced daily by courier services (e.g., Federal Express, United Parcel Service, and Overnight United States Postal Service), local trucking companies, and demand responsive transportation services, just to name a few. These types of services have experienced tremendous growth in recent years. For example, both United Parcel Service and Federal Express have annual revenue of well over \$10 billion, and the dial-a-ride service for the disabled and handicapped is today a \$1.2 billion industry (Palmer, Dessouky, and Abdelmaguid 2004). However, congestion and variability in demand and travel times affects these industries on three major service dimensions: travel time, reliability, and cost (Meyer 1996). Therefore, there is a need to develop routing and scheduling tools that directly account for the uncertainty. In this paper, we focus on the uncertainty in demand.

Generally speaking, current methods to address the uncertainty in the Vehicle Routing Problem (VRP) consider one of the following strategies (or both): they either make strong assumptions regarding the distribution of the uncertain parameters, or the methods approximate the stochastic VRP with a much larger deterministic model in which the uncertainty is represented through scenarios (Bertsimas and Simchi-Levi 1996; Gendreau, Laporte, and Seguin 1996). The objective of these methods is to obtain a solution that minimizes the expected value, or to analyze the performance of a routing policy in expected value or worst case. The resulting solution is potentially sensitive to the actual data that occurs in the problem. Considering that the VRP solution in a given application will only face a single realization of the uncertainty, a reasonable goal is to obtain a *robust* solution, i.e. a solution that is *good* for all possible data uncertainty.

In this paper, we consider the capacitated VRP (CVRP) with uncertain demand on a set of fixed demand points. We use the robust optimization methodology introduced by Ben-Tal and Nemirovski (1998) to formulate a new problem for the VRP with demand uncertainty, the Robust Vehicle Routing Problem (RVRP). The optimal solution for this problem is the

route that optimizes the worst case value over all data uncertainty. The expectation is that such a solution would be efficient in its worst case and thus efficient for every possible outcome of data. The robust version of a minimization problem under uncertainty is obtained by a combined minimization-maximization problem, which is typically more difficult to solve than only the minimization part. In our robust optimization methodology, we are interested in introducing uncertainty in demand for the CVRP in such a way as to obtain a RVRP which can be solved efficiently. That is to say that it is not significantly more difficult than solving the deterministic CVRP.

A natural method to address demand uncertainty is to reserve vehicle capacity to be able to adapt to cases when the realized demand is greater than the expected demand. In fact, if there is abundant vehicle capacity, such as in the uncapacitated VRP, the optimal routing solution can easily accommodate changes in the demand levels. However, in capacitated cases where the vehicle capacity is slightly greater than the expected demand, the interesting problem is how to manage the extra vehicle capacity to distribute slack among routes to better address the demand uncertainty. The RVRP distributes this slack by finding a routing solution at minimum cost that satisfies all possible demand realizations.

The structure of the paper is as follows. We discuss the relevant literature in the next section. In Section 3 we present the derivations of the RVRP formulations for problems with demand uncertainty and show that for the Miller-Tucker-Zemlin (MTZ) formulation and demand uncertainty sets constructed from combinations of scenarios the resulting RVRP is another instance of a CVRP. We present our computational results in Section 4. These include a comparison of the robust and deterministic solution on a well-known suite of CVRP problems (Augerat et al. 1995), a comparison on a family of clustered instances, and verifying that the robust solution better addresses the demand uncertainty than a uniform distribution of unused vehicle capacity. We finish the paper with some conclusions in Section 5.

2 Literature Review

Problems where a given set of vehicles with finite capacity have to be routed to satisfy a geographically dispersed demand at minimum cost are known as Vehicle Routing Problems (VRP). This class of problems was introduced by Dantzig and Ramser (1959) and since has lead to a considerable amount of research on the VRP itself and its numerous extensions and applications. General surveys of vehicle routing research can be found in Toth and Vigo (2002), Fisher (1995), and Laporte and Osman (1995). The VRP is known to be NP-Hard (Lenstra and Rinnooy Kan 1981), but nevertheless, there is considerable work on developing exact solution procedures, see for instance (Lysgaard et al. 2004; Baldacci et al. 2004; Ralphs et al. 2003; Fukasawa et al. 2006).

The most studied area in the stochastic vehicle routing problem literature has been the VRP with stochastic demands (VRPSD), and with stochastic customers (VRPSC). A major contribution to the study of the VRPSD comes from Bertsimas (1992). This work illustrated the a priori method with different recourse policies (re-optimization is allowed) to solve the VRPSD and derived several bounds, asymptotic results and other theoretical properties. Bertsimas and Simchi-Levi (1996) surveyed the development in the VRPSD with an emphasis on the insights gained and on the algorithms proposed. Besides the conventional stochastic programming framework, a Markov decision process for single stage and multistage stochastic models were introduced to investigate the VRPSD in Dror et al. (1989) and Dror (1993). More recently, a re-optimization type routing policy for the VRPSD was introduced by Secomandi (2001).

The VRPSC, in which customers with deterministic demands and a probability p_i of being present, and the VRP with stochastic customers and demands (VRPSCD), which combines the VRPSC and VRPSD, first appeared in the literature of Jézéquel (1985), Jaillet (1987) and Jaillet and Odoni (1988). Bertsimas (1988) gave a more systematic analysis and presented several properties, bounds and heuristics. Gendreau et al. (1995, 1996) proposed the first exact solution, an L-shaped method, and a meta-heuristic, a tabu search for the VRPSCD.

Compared with stochastic customers and demands, the VRP with stochastic service

time and travel time (VRPSSTT) has received less attention. Laporte et al. (1992) proposed three models for the VRPSSTT: chance constrained model, 3-index recourse model and 2-index recourse model, and presented a general branch-and-cut algorithm for all 3 models. The VRPSSTT model was applied to a banking context and an adaptation of the savings algorithm was used in the work of Lambert et al. (1993). Jula et al. (2006) developed a procedure to estimate the arrival time to the nodes in the presence of hard time windows. In addition, they used these estimates embedded in a dynamic programming algorithm to determine the optimal routes.

Different solution strategies for the VRPSD will vary in their allocation of the capacity that is unused in the expected demand scenario. In situations where this unused capacity or slack is small, how it is used will be determinant in being able to cope with the uncertainty. A few methods developed for the deterministic VRP have focused on how to distribute the vehicle capacity among routes. For instance, Daganzo (1988) proposes the use of a consolidation center as a strategy to better manage vehicle capacity. In Charikar et al. (2001) the authors introduce an approximation algorithm that is no worse than 5 times the optimal solution. This algorithm begins with a solution in which every vehicle has half of its capacity unused. This slack is then used to improve the routes through a matching algorithm. Branke et al. (2005) show that by appropriate use of the slack through waiting at strategic locations can increase the probability of meeting additional demand. Our work departs from these prior results as we consider a different problem domain: a standard CVRP with no transshipment nodes and with a small capacity to demand ratio. When it comes to the stochastic VRP, Zhong et al. (2004) considered a VRP where customer locations and demands are uncertain. They developed a two-stage model that uses the capacity that is not assigned in the first stage to adapt to the demand uncertainty in the second stage. The first stage creates “core areas” to be serviced, and after the demand is realized the recourse actions involve how to route in these areas allowing for exchanging demand nodes on the “flex-zones” between core areas. They showed that keeping customers near the depot unassigned is a good strategy for balancing the workload due to daily demand variations. Although our work develops

an a-priori routing strategy with no recourse, we also notice that having customers near the depot facilitates the creation of efficient robust routes.

In this paper we address the demand uncertainty in the VRP using robust optimization. The current robust optimization research is closely related to robust control theory; see for example Zhou et al. (1996). These ideas are ported to a mathematical programming context beginning with the work by Ben-Tal and Nemirovski (1998, 1999) where the authors formulate the robust optimization problems of linear programs, quadratic programs, and general convex programs. Independently El-Ghaoui et al. (1998) studied the same robust optimization counterpart for semidefinite programming problems. More recently, this approach has been extended to integer programming problems (Bertsimas and Sim 2003). Robust solutions have the potential to be viable solutions in practice, since they tend not to be far from the optimal solution and significantly outperform the optimal solution in the worst case (Goldfarb and Iyengar 2003; Bertsimas and Sim 2004).

The general approach of robust optimization is to optimize against the worst instance that might arise due to data uncertainty by using a min-max objective. The resulting solution from the robust counterpart problem is insensitive to the data uncertainty, as it is the one that minimizes the worst case, and therefore is *immunized* against this uncertainty. The robust optimization methodology assumes the uncertain parameters belong to a bounded uncertainty set. For fairly general uncertainty sets, the resulting robust counterpart can be similar to the original problem, and therefore can have comparable complexity. For example, a linear program with uncertain parameters belonging to a polyhedral uncertainty set has a robust problem which is an LP whose size is polynomial in the size of the original problem (Ben-Tal and Nemirovski 1999). An important question is how to formulate a robust problem that is not more difficult to solve than its deterministic counterpart.

3 RVRP Formulations

There exist a number of different VRP formulations and since each would yield a different RVRP, it is important to identify a VRP formulation that leads to a RVRP that is not too difficult to solve. In addition to the VRP formulation, the form of the uncertainty sets considered also influences the resulting RVRP and the difficulty in solving it.

In this section, we first identify the deterministic VRP formulation and demand uncertainty sets that will be used and then we present the derivation for the RVRP.

3.1 Identifying the VRP Formulation

In addition to the problem size, the difficulty in solving a problem is influenced by three aspects: the problem data, the problem formulation and the solution procedure. For instance, the observed run-times of a fixed IP solver show different behavior as we vary the VRP formulations (Ordóñez et al. 2005). In addition, the fixed general IP solver was most efficient in solving the Miller-Tucker-Zemlin (MTZ) formulation than other arc-based VRP formulations considered in that study for a wide range of problem parameters.

Another important criterion in identifying a suitable formulation for our robust optimization framework is the nature of the formulation with respect to uncertain parameters. Since we are interested in introducing uncertainty in demand, when we consider the parts of the formulation related to demand, the MTZ formulation has constraints in the form of inequalities. In the robust optimization methodology, it is preferable to have inequality constraints involving uncertain parameters than equality constraints, since it is more difficult to satisfy equalities for all values of the uncertainty. In fact Ben-Tal et al. (2003) shows that even for simple linear programs, if there are uncertain parameters in equality constraints the robust counterpart problem can be NP-hard.

The MTZ formulation of the CVRP follows: it considers the problem of routing at minimum cost a uniform fleet of K vehicles, each with capacity C , to service geographically dispersed customers, each with a deterministic demand that must be serviced by a single

vehicle. Let V be the set of n demand nodes and a single depot, denoted as node 0. Let d_i be the demand at each node i . We consider the fully connected network, and denote the deterministic travel time between node i and node j by c_{ij} . The arc-based model considers integer variables x_{ij} which indicate whether a vehicle goes from node i to node j or not. In addition, the MTZ formulation includes continuous variables u_i for every $i \in V \setminus \{0\}$ that represent the flow in the vehicle after it visits customer i . The constraints (1.2-1.5) are routing constraints and the constraints (1.6) and (1.7) impose both the capacity and connectivity of the feasible routes.

$$\begin{aligned}
(\text{CVRP}) \quad & \min \sum_{i \in V} \sum_{j \in V} c_{ij} x_{ij} & (1.1) \\
& \text{s.t.} \quad \sum_{i \in V} x_{ij} = 1 & j \in V \setminus \{0\} & (1.2) \\
& \sum_{j \in V} x_{ij} = 1 & i \in V \setminus \{0\} & (1.3) \\
& \sum_{i \in V} x_{i0} = K & (1.4) & (1) \\
& \sum_{j \in V} x_{0j} = K & (1.5) \\
& u_j - u_i + C(1 - x_{ij}) \geq d_j \quad i, j \in V \setminus \{0\}, i \neq j & (1.6) \\
& d_i \leq u_i \leq C & i \in V \setminus \{0\} & (1.7) \\
& x_{ij} \in \{0, 1\} & i, j \in V & (1.8) .
\end{aligned}$$

Notice that the uncertain demand d_i appears by itself and only on constraints (1.6) and (1.7). However, the lower bound on constraint (1.7) is implied from (1.6), the fact that every node is visited, and that $u_i \geq 0$ for all $i \in V \setminus \{0\}$. We will therefore only consider the uncertainty in (1.6) and replace all d_i with 0 in constraint (1.7).

Although the nature of the MTZ formulation is the preferred one with respect to uncertain parameters for our robust optimization framework, there is a caveat. This formulation is known to be very sensitive to the problem parameters which is most likely due to the large initial LP gap of the formulation. However, it is possible to improve this gap by adopting the lifting techniques proposed by Desrochers and Laporte (1991).

3.2 Uncertainty in Demand

In this subsection, we consider the demand parameter d to be uncertain and to belong to a bounded set U_D . We consider uncertainty sets which are constructed as deviations around an expected demand value d^0 . The possible deviation directions from these nominal values are fixed and identified by scenario vectors, $d^k \in \mathfrak{R}^n$, where n is the number of nodes. The scenario vectors are allowed to have negative deviation values. For a given number of scenario vectors, s , the general uncertainty set U_D is a linear combination of the scenario vectors with weights $y \in \mathfrak{R}^s$ that must belong to a bounded set $y \in Y$:

$$U_D = \left\{ d \mid d^0 + \sum_{k=1}^s y_k d^k, y \in Y \right\}$$

In particular, we consider the following three sets for Y :

$$\begin{aligned} \text{convex hull} \quad Y_1 &= \left\{ y \in \mathfrak{R}^s \mid y \geq 0, \sum_{k=1}^s y_k \leq 1 \right\} \\ \text{box} \quad Y_2 &= \{ y \in \mathfrak{R}^s \mid \|y\|_\infty \leq 1 \} \\ \text{ellipsoidal} \quad Y_3 &= \{ y \in \mathfrak{R}^s \mid y^T Q y \leq 1 \} \text{ ,} \end{aligned}$$

where the ellipsoidal set is defined for some given positive definite matrix Q , for example $Q = I$. We refer to the uncertainty set formed by considering the combination set Y_i as U_{Di} for $i = 1, 2, 3$. Note that if $s = n$ and the scenario vectors d^k correspond to the coordinate axis, then Y_2 leads to $U_D = d^0 + \{d \mid \|d\|_\infty \leq 1\}$ and Y_3 to $U_D = d^0 + \{d \mid d^T Q d \leq 1\}$ the full dimensional box and ellipse centered at d^0 , respectively. We will show that for these three sets U_{Di} the resulting RVRP problem is an instance of CVRP.

3.3 Robust VRP formulation

We now propose the robust counterpart problem RVRP for CVRP with demand belonging to an uncertainty set U_D . Recall that we consider the problem only with uncertainty in constraint (1.6) with constraint (1.7) equal to $0 \leq u \leq C$.

The robust VRP finds the optimal route that satisfies all possible demand outcomes, in other words the problem has to identify routes x_{ij} and vehicle usage u_i such that

$$u_j - u_i + C(1 - x_{ij}) \geq d_j \quad \forall d \in U_D \quad i, j \in V \setminus \{0\}, i \neq j \quad (1.9).$$

We can therefore state the RVRP. This problem minimizes objective (1.1), subject to constraints (1.2), (1.3), (1.4), (1.5), (1.7), (1.8), (1.9). If we substitute in the definition of the uncertainty set U_D , we can write the robust constraint (1.9) as the following inequality

$$u_j - u_i + C(1 - x_{ij}) - d_j^0 \geq \sum_{k=1}^s y_k d_j^k \quad \forall y \in Y \quad i, j \in V \setminus \{0\}, i \neq j \quad (1.10)$$

For given decision variables x and u we refer to the right hand side of the above inequality as $\phi_{ij}(x, u) = u_j - u_i + C(1 - x_{ij}) - d_j^0$ for $i, j \in V \setminus \{0\}, i \neq j$. Then, to enforce that the above inequality holds for all $y \in Y$ it suffices to enforce it for $\sup_{y \in Y} \sum_{k=1}^s y_k d_j^k = \sup_{y \in Y} y^T D_{j\bullet}$. Here we denote by $D = [d^1 \dots d^s] \in \mathfrak{R}^{n \times s}$ the matrix of scenario vectors and $D_{j\bullet} = (d_j^1, \dots, d_j^s)^T$ the j -th row of D as a column vector. Let us also denote e as the column vector of all 1 of appropriate dimension.

Proposition 1 *Under uncertainty set U_{D1} , the robust counterpart is obtained by replacing constraint (1.6) in CVRP with the constraint below (1.11). We refer to the resulting RVRP as RVRP₁.*

$$u_j - u_i + C(1 - x_{ij}) \geq d_j^0 + \max\{\max_k d_j^k, 0\} \quad i, j \in V \setminus \{0\}, i \neq j \quad (1.11)$$

Proof: Using the definition of Y_1 we can write $\sup_{y \in Y_1} y^T D_{j\bullet}$ and its dual as the following pair of LPs:

$$\begin{array}{ll} \text{(Primal)} & \max \quad y^T D_{j\bullet} \\ & \text{s.t.} \quad e^T y \leq 1 \\ & \quad y \geq 0 \\ \text{(Dual)} & \min \quad \theta \\ & \text{s.t.} \quad \theta e \geq D_{j\bullet} \\ & \quad \theta \geq 0. \end{array}$$

From weak duality, the condition $\phi_{ij}(x, u) \geq \sup_{y \in Y_1} y^T D_{j\bullet}$ is equivalent to having $\phi_{ij}(x, u) \geq \theta$ for some dual feasible θ . This means that $\phi_{ij}(x, u) \geq 0$ and $\phi_{ij}(x, u) \geq d_j^k$ for $k = 1, \dots, s$. Combining these conditions together for all $\phi_{ij}(x, u)$ gives (1.11). \square

Proposition 2 Under uncertainty set U_{D_2} , the robust counterpart is obtained by replacing constraint (1.6) in CVRP with the constraint below (1.12). We refer to the resulting RVRP as $RVRP_2$.

$$u_j - u_i + C(1 - x_{ij}) \geq d_j^0 + \sum_k |d_j^k| \quad i, j \in V \setminus \{0\}, i \neq j \quad (1.12)$$

Proof: Using the definition of Y_2 we can write $\sup_{y \in Y_2} y^T D_{j\bullet}$ and its dual as the following pair of LPs:

$$\begin{array}{ll} \text{(Primal)} & \max \quad y^T D_{j\bullet} \\ & \text{s.t.} \quad y \leq e \\ & \quad \quad y \geq -e \\ \text{(Dual)} & \min \quad e^T(\alpha + \beta) \\ & \text{s.t.} \quad \alpha - \beta = D_{j\bullet} \\ & \quad \quad \alpha, \beta \geq 0. \end{array}$$

It is simple to verify that the optimal solution to the dual problem will satisfy $\alpha_k^* + \beta_k^* = |d_j^k|$ for every $k = 1, \dots, s$. Therefore the dual optimal objective value is $\sum_{k=1}^s |d_j^k|$. Enforcing the robust feasibility condition on $\phi_{ij}(x, u)$ with the above optimal dual objective value we obtain (1.12). \square

Proposition 3 Under uncertainty set U_{D_3} , the robust counterpart is obtained by replacing constraint (1.6) in CVRP with the constraint below (1.13). We refer to the resulting RVRP as $RVRP_3$.

$$u_j - u_i + C(1 - x_{ij}) \geq d_j^0 + \sqrt{D_{j\bullet}^T Q^{-1} D_{j\bullet}} \quad i, j \in V \setminus \{0\}, i \neq j \quad (1.13)$$

Proof: Using the definition of Y_3 we have that $\sup_{y \in Y_3} y^T D_{j\bullet} = \max y^T D_{j\bullet} : y^T Q y \leq 1$. From the KKT optimality conditions we have that the optimal solution to this problem is $y^* = \frac{1}{\sqrt{D_{j\bullet}^T Q^{-1} D_{j\bullet}}} Q^{-1} D_{j\bullet}$. When we plug this optimal solution into the robust feasibility condition $\phi_{ij}(x, u) \geq (y^*)^T D_{j\bullet}$, we obtain (1.13). \square

For the three RVRPs with demand uncertainty studied, the only change from the original CVRP formulation is an increase in the demands that appear in (1.6). Since the deviation

vectors, d^k , are fixed, each of the RVRPs is an instance of the CVRP. We can therefore make use of the efficient exact algorithms in the literature to solve the robust problems. We note that the demand parameters used in the RVRPs are at least as big as the deterministic demand parameters, thus the RVRPs are typically more capacity constrained than the corresponding deterministic problem. We note this because it has been observed in practice that solving CVRP becomes harder as the problem is more capacity constrained. Thus, although the RVRPs are instances of CVRP, in practice solving RVRPs is likely to be more difficult than solving the deterministic versions. Lastly note that, depending on the nature of the scenario vectors, RVRPs may result in infeasible problems even though the deterministic CVRP is feasible.

For different types of demand uncertainty sets $d \in U$, the key step in the derivation of the RVRP is to compute $\sup_{d \in U} d_j$ and substitute this value for the right hand side of equation (1.9). This can be done for different uncertainty sets than considered here. We do not pursue these formulations here for simplicity, since many require additional constraints and variables making the resulting robust problem not a CVRP that may necessitate a specialized solution procedure.

Since the robust formulations with uncertainty in demand, RVRP₁, RVRP₂, and RVRP₃ are an instance of CVRP, it is possible to introduce uncertainty in travel time in addition to the uncertainty in demand by using the approach proposed by Bertsimas and Sim (2003) for integer programs with uncertain cost coefficients. The authors consider a box uncertainty set for the cost coefficients with an additional restriction on the number of cost coefficients that vary. They show that the optimum solution of the robust counterpart can be obtained by solving a polynomial number of nominal problems with modified cost coefficients. Since the RVRPs for our proposed uncertainty sets are an instance of a general integer program, formulating and solving the robust counterparts with independent uncertainty in both demand and travel time is a straightforward application of this methodology.

4 Experimental Analysis

In this section, we first present performance measures that will be used to compare robust and deterministic solutions. We then analyze the trade-offs of robust solutions on instances from the literature and families of clustered instances. We finish the section verifying that the robust solution provides better protection to the demand uncertainty than a uniform distribution of the unused capacity among the vehicles. We present computational results only for the convex hull uncertainty set. Similar results are obtained for box and ellipsoidal uncertainty sets but we omit them for space considerations.

Our experiments require the solution of the deterministic and robust versions of the routing problem, since all are instances of CVRP we use an efficient exact solution procedure for generic CVRP. We use the branch-and-cut based VRP solver in the open source SYMPHONY library due to Ralphs et al. (2003), available on-line at <http://branchandcut.org/VRP>. All experiments are carried out with a runtime limit of one hour on a Dell Precision 670 computer with a 3.2 GHz Intel Xeon Processor and 2 GB RAM running Red Hat Linux 9.0.

4.1 Performance measures

The first performance measure, the ratio r_{rd} , quantifies the relative extra cost of the robust with respect to the cost of the deterministic. It is given by $r_{rd} = \frac{z_r - z_d}{z_d}$ where z_d is the optimal objective function value of the deterministic CVRP and z_r is the optimal objective value of the robust counterpart. This ratio gives information on how much extra cost we will incur if we want to implement the robust to protect against the worst case realization of the uncertainty, instead of implementing the deterministic. Note that the calculation of the ratio requires solving two instances of CVRP.

The second performance measure considers the effect of the solutions on the demand when it is subject to demand uncertainty. The ratio r_{ud} is the relative unsatisfied demand for the deterministic solution when it faces its worst case demand. It is given by $r_{ud} = \frac{z_{ud}}{\sum_{i \in V} d_i^0}$ where the numerator z_{ud} is the maximum unsatisfied demand that can occur if the optimal

deterministic solution is used. The denominator is the total demand of the deterministic case and it is assumed that the deterministic problem is feasible. To obtain z_{ud} , we fix the routing variables to the deterministic optimal solution and maximize the unmet demand by varying the demand outcome within the demand uncertainty set. Note that the calculation of this ratio requires solving only one instance of the CVRP.

4.2 Robust versus deterministic on standard problems

Our first set of experiments address problem set A (Random Instances), set B (Clustered Instances), and set P (Modified Instances from the literature) of the CVRP suite of problems by Augerat et al. (1995). The instances range from 15 to 100 customers. We modified these instances to include demand uncertainty. We allow each demand parameter to further increase up to a fixed percentage of the deterministic value. We randomly generate a total of 5 scenarios within the allowed percent deviation for the demand uncertainty set. More specifically, we use the following values of percent deviation in demand parameters: 5, 10, 15, and 20.

Table 1 shows the results based on the performance measures r_{ud} for the percent unmet demand ratio and r_{rd} for the percent cost ratio of the solutions, where “No” indicates the number of the instance, “T” indicates the percent tightness ratio of the instance which is defined as the ratio of the total expected demand to total vehicle capacity, “IN” indicates infeasible instance, and “NA” indicates that an optimal solution could not be found within the 1 hour runtime limit.

The first observation is that since the original instances are already tight (they have a percent tightness ratio between 81% and 99%), the robust counterparts run quickly into infeasibility as the percent deviation of uncertainty increases even though the deterministic CVRPs are feasible and could be solved to optimality. In almost all of the instances with a tightness ratio greater than 90%, the cost ratio could not be calculated for the percent deviation values 15% and 20% since either the robust counterpart became infeasible or the runtime limit was reached. Recall that the robust instances are more capacity constrained

Table 1: Augerat et al. (1995) Sets

No	Set A								Set B								Set P										
	T	Unmet Demand				Cost Ratio				T	Unmet Demand				Cost Ratio				T	Unmet Demand				Cost Ratio			
		5	10	15	20	5	10	15	20		5	10	15	20	5	10	15	20		5	10	15	20	5	10	15	20
1	82	0.5	2.5	4.8	6.9	0	1.5	4.2	9.3	82	0	0.7	2.6	4.5	0	1.3	1.6	3.3	87	0.5	1.2	2.2	3.3	0	2.2	5.1	IN
2	89	0.2	1.7	3.5	5.9	0	2.1	6.5	IN	91	1	2.5	4.4	6.5	0	0.1	5.5	IN	96	0.5	2.3	5	8.2	0	IN	IN	IN
3	90	0.3	1.8	4	6.5	0	2.7	7.1	IN	87	0.7	2.3	4.4	6.7	0.1	2.8	3.2	4.4	96	0.8	2.2	4.8	7.5	0	IN	IN	IN
4	92	0	0.3	1.6	3.2	0	1.5	4.6	IN	85	0.4	2	4.2	7.3	0	0.1	1.7	1.9	93	0	0	0.9	3.4	0	0	0	IN
5	88	0.8	2.1	4	6.3	0	1.8	4.5	6.3	88	1.2	2.7	4.6	6.7	0	2.2	2.6	6.4	96	0.9	2.4	3.8	5.9	0	3.2	IN	IN
6	81	0.2	1.6	3.3	4.8	0	2.7	4.8	5.4	94	0.9	2.4	4.3	6.7	0	4.2	IN	IN	93	0.7	1.9	3.3	4.9	14.9	IN	IN	IN
7	95	0.7	2.4	4.5	6.8	0.1	NA	IN	IN	86	1.1	3.5	6.4	9.4	0	1.5	4.9	5.5	97	1.2	3.4	5.8	8.2	2.8	IN	IN	IN
8	96	1.4	3.8	6.4	9	0	IN	IN	IN	91	1.2	3.6	6	8.5	0	4.6	8.5	IN	88	0.3	1.2	2	3.1	0	0.7	2.4	IN
9	95	1	3.1	5.2	7.9	0	1.5	IN	IN	97	0.9	3.1	5.3	8.1	0	IN	IN	IN	92	1.2	3.3	5.5	7.6	0.6	2	IN	IN
10	87	1.3	4.2	7.3	10	0	0.4	2.3	5.1	98	1.8	4.4	6.9	9.3	0	IN	IN	IN	90	0.5	2	4.1	6.1	0.5	2	NA	IN
11	95	0.9	2.2	3.8	5.8	0	1.5	IN	IN	87	0.6	1.5	2.8	4.2	0	0.5	1.3	1.9	99	1.5	3.9	6.7	9.3	IN	IN	IN	IN
12	98	1	3.7	6.4	9.1	0	IN	IN	IN	91	1.2	3	5.3	7.4	NA	2.7	NA	IN	95	0.7	2.1	4.3	6.7	NA	IN	IN	IN
13	90	0.8	2.4	4	6	NA	NA	NA	IN	97	1.2	3.2	5.8	8.7	0	IN	IN	IN	97	1.1	3.2	5.3	7.7	NA	IN	IN	IN
14	86	1.4	3.4	5.7	8.1	0	4.2	6.8	8	86	0.7	2	3.7	5.7	0	0.7	1.5	3.2	87	0.4	2	4.3	6.9	NA	NA	NA	NA
15	89	1	2.6	4.5	6.6	0	NA	NA	IN	88	1.1	3	4.9	6.9	0	3	3.4	5.8	81	0.5	1.9	3.6	5.3	NA	NA	NA	NA
16	94	1.1	3.3	5.7	8.2	0.5	3.7	IN	IN	99	2	5	8.1	11	IN	IN	IN	IN	90	0.8	2.9	5.1	7.7	NA	NA	NA	IN
17	95	0.4	1.9	4.5	7	NA	NA	IN	IN	89	1.2	3.1	5.3	7.5	NA	NA	NA	NA	99	1.6	4.4	7.2	10	NA	IN	IN	IN
18	93	1.3	3.4	5.8	8.5	0.1	2.8	IN	IN	92	1.1	3	5.1	7.4	NA	NA	NA	IN	94	0.5	1.7	3.7	5.9	NA	NA	IN	IN
19	92	0.9	2.1	3.8	5.8	NA	NA	NA	IN	97	1.2	3.3	5.6	8	0	IN	IN	IN	94	0.8	2.3	4	5.9	NA	NA	IN	IN
20	98	1.7	4.4	7.2	10	NA	IN	IN	IN	95	1.4	3.6	5.9	8.4	NA	IN	IN	IN	93	0.6	1.8	3.3	5.3	NA	NA	IN	IN
21	91	1.3	3.6	6.1	8.6	NA	4.7	NA	IN	90	1	2.5	4.4	6.5	NA	NA	NA	IN	97	0.5	2.3	4.4	6.6	NA	IN	IN	IN
22	97	1.2	3.1	5.4	7.7	NA	IN	IN	IN	93	1.6	3.9	6.4	9.1	NA	NA	IN	IN	97	0.8	2.4	5	7.7	0.5	IN	IN	IN
23	93	1.1	3.3	5.6	8.1	NA	NA	IN	IN	93	1.4	3.7	6.2	8.9	NA	NA	IN	IN	97	0.9	2.5	5.1	7.9	NA	IN	IN	IN
24	94	1.2	3.5	5.9	8.4	NA	NA	IN	IN									91	0	0.9	2.9	5	0	0.4	0.6	IN	
25	97	1.3	3.8	6.3	8.9	NA	IN	IN	IN																		
26	93	0.8	2.5	4.5	6.8	NA	NA	IN	IN																		
27	94	0.7	2.5	4.4	6.6	NA	NA	IN	IN																		

due to increased demand in the data and empirical observations have shown that CVRPs become more difficult to solve in practice as problems are more capacity constrained.

To compare the ratios in each problem set to analyze the average behavior of the robust and the deterministic, we count the number of times the cost ratio is not worse than the unmet demand ratio. When the increase in the unmet demand on the worst case scenario for the deterministic solution is greater than the increase in cost for the robust solution on the nominal case, we say the robust solution is preferable. We understand that the magnitude for comparison of the ratios is application and scenario dependent. For simplicity, we weight both ratios equally in our comparisons in this paper. In set A, robust is preferable in 27 cases out of 42 possible comparisons; in set B, it is 39 cases out of 45; and in set P, it is 16 out of 23. We see that the robust outperforms the deterministic the most in set B where the instances are clustered; that is the additional cost of implementing the robust is no worse than the amount of unmet demand incurred in the worst case if the deterministic was implemented.

A close analysis of the solutions revealed that the success of the robust for the clustered instances is directly related to the distribution of the unused vehicle capacity slack over the network. When the realized demand is greater than the expected demand, the additional slack helps address the increase in demand. If there is not enough slack in the vehicle for the additional demand, then the robust will route the vehicles differently. In this case, if there is a vehicle nearby with enough slack, then this demand can be serviced by this latter vehicle with only a slight cost increase. However, if a vehicle with enough slack is far from this extra demand, then the robust solution that can satisfy this demand will be significantly more expensive.

Consider for example the deterministic and robust optimal solutions for instance number 3 from set B depicted in Figure 1. The numbers next to the routes indicate the total expected demand, d^0 serviced on each route. The slack of each route is the vehicle capacity, 100, minus this total expected demand. We label the customers according to the deterministic route that services them, keeping the label for the robust route to emphasize the change in solution.

We see that five units of slack from the vehicle which services 88 units of expected demand in the deterministic is distributed in the robust to the other vehicles without significantly increasing the total cost, even though it affected three other routes. Also note that the route of the vehicle which services 56 units of total demand in the deterministic stays the same in the robust since there is already enough slack in this vehicle to accommodate the increase in the demand due to worst case realization of the uncertainty. Overall, for this particular case, the increase in the cost due to robust is less significant than the total unmet demand that occurs in the worst case realization of the uncertainty for the deterministic.

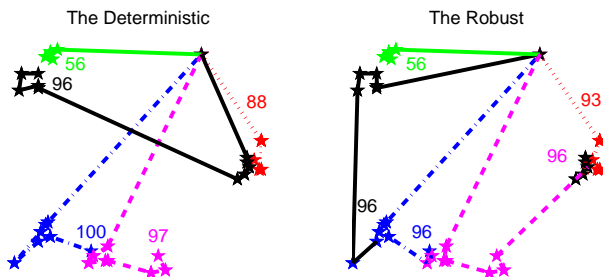


Figure 1: Deterministic and Robust solutions for Augerat et al. (1995) Set B Instance 3. No. of vehicles 5, capacity 100, value next to route is the deterministic demand served.

On the other hand, when the network structure does not allow an easy distribution of slack, then the robust may result in a poor performance as depicted in Figure 2, instance number 1 from the set B. The vehicles have again the identical capacity of 100. In this case, the slack in the vehicle which services 38 units of total demand in the deterministic is costly to distribute in the robust to the other vehicles since the route of that vehicle is relatively far from the routes of the others.

These examples show that the structure of the network plays a key role in determining optimal routes and thus the distribution of the unused capacity and its impact on the success of a robust solution. The instances in the Augerat et al. (1995) suite of problems suggest that when the network is clustered optimal solutions will have some vehicles close which could share unused capacity at a low cost favoring a robust solution.

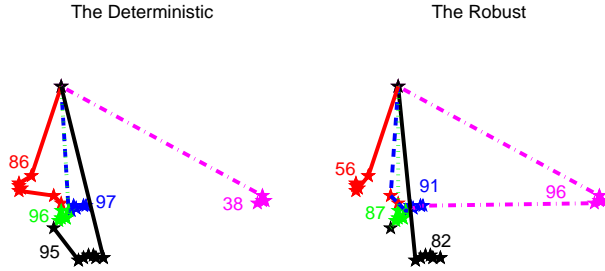


Figure 2: Deterministic and Robust solutions for Augerat et al. (1995) Set B Instance 1. No. of vehicles 5, capacity 100, value next to route is the deterministic demand served.

4.3 Robust versus deterministic on family of clustered instances

To validate our findings and to generalize them with respect to the structure of the network, we randomly generate instances with 4 vehicles of capacity 1500 and 49 customers with uniform demand of 100, in three different problem sets. In each set, there are 4 clusters of customers. First of all, we consider points which are on the circle of a given radius R , centered at a depot, and we randomly select a point on that circle to be the center of a cluster (see Figure 3). Then we generate customers for that cluster within the circle with a given radius r . We also use a measure for clustering for our instances, r_c , which is given by:

$r_c = \frac{R}{r}$. We fix the value of $r = 20$ and consider the following values for R :

R	0	$2r$	$4r$	$6r$	$8r$
-----	---	------	------	------	------

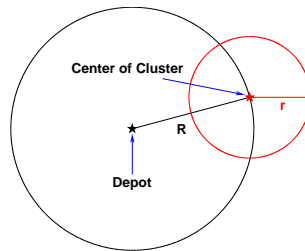


Figure 3: Cluster generation in Random Sets

When $r_c = 0$ all the clusters are centered at the depot and the instance becomes random with no clustering effect; however as r_c increases, clusters separate from each other. In problem sets 1 and 2, the three clusters have 13 customers and the fourth one has 10. Note that a vehicle can service up to 15 customers. The reason for this selection is that, in clustered instances where each cluster will be serviced by only one vehicle, there will be one vehicle with relatively more slack, namely the one servicing the fourth cluster. The only difference between sets 1 and 2 is that in the latter as we increase r_c , we always keep the fourth cluster centered at the depot. This serves the purpose of having a random zone around the depot and some clusters far from the depot. In set 3, we make the random zone denser by increasing the number of customers in the fourth cluster to 25 and decreasing the one for the others to 8. Figures 4, 5, and 6 display the results of the three sets for percent unmet demand ratio r_{ud} and percent cost ratio r_{rd} as a function of percent clustering ratio r_c for different values of percent deviations of the uncertainty set. Each data point on the figures is an average of 30 instances.

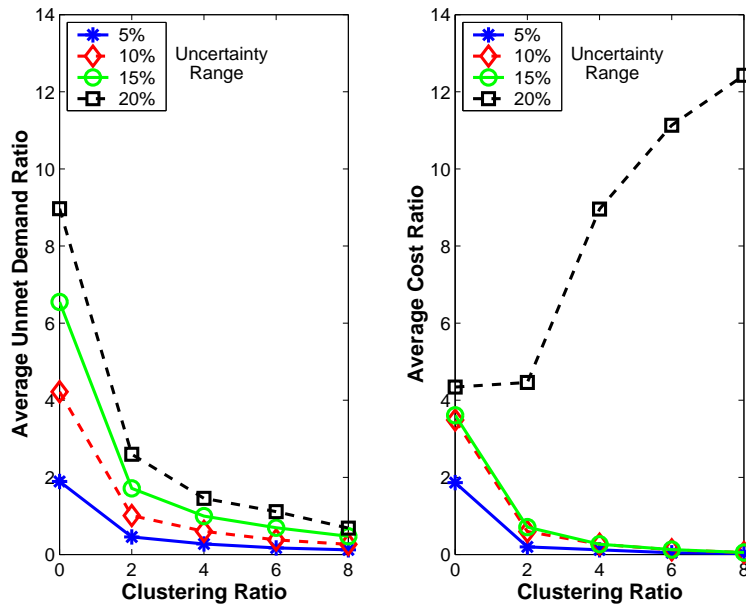


Figure 4: Comparison of Deterministic and Robust solutions for Random Set 1

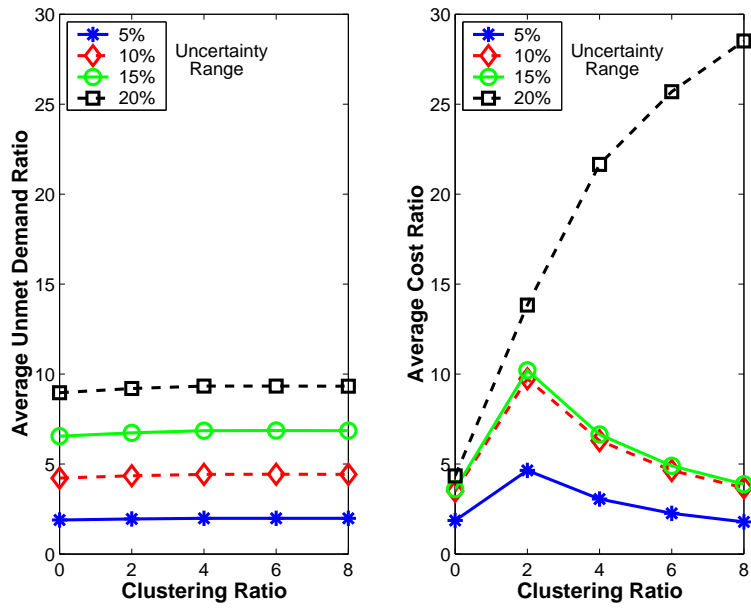


Figure 5: Comparison of Deterministic and Robust solutions for Random Set 2

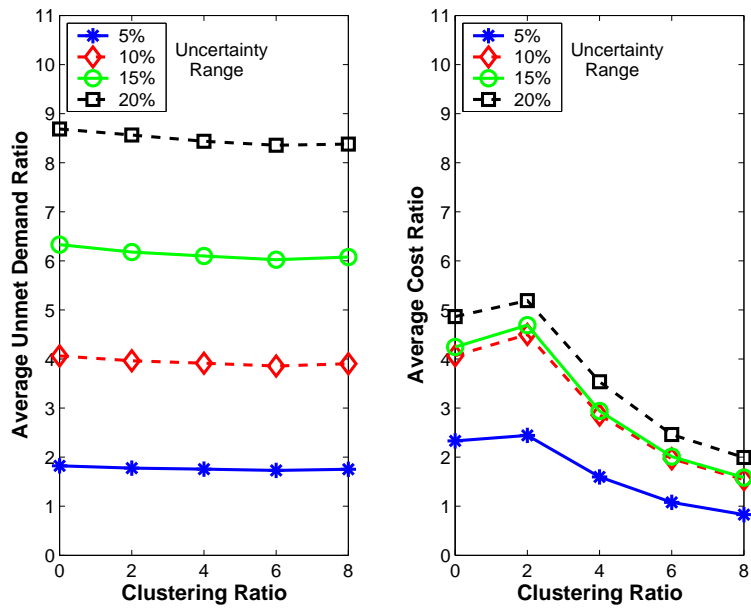


Figure 6: Comparison of Deterministic and Robust solutions for Random Set 3

The results of set 1 suggest that both the deterministic and the robust benefit from clustering. For percent deviation of 5% and 10%, both results are comparable, for 15% the robust is better, and for 20% the robust is worse when $r_c \geq 2$. In fact as the uncertainty increases, we would expect the robust to outperform the deterministic. The reason for this odd behavior is the distribution of the slack in the network. When the instances are clustered for the bigger values of r_c , each cluster is serviced only by one vehicle in the deterministic. In case of high uncertainty such as 20% deviation, if the total demand of a cluster exceeds the vehicle capacity then another vehicle has to be routed to this cluster by the robust. When these vehicles are not close, the robust results in a large travel cost. The network structures with pure clusters as in set 1 therefore do not allow a good distribution of slack on the average and the robust is not convenient for high uncertainties.

When we look at the results of set 2, as before we see the same phenomenon in the increase of the cost ratio for the robust with 20% deviation. However, clustering helps only after $r_c > 2$. The reason is due to the random zone around the depot. When $r_c \leq 2$, the circles of clusters intersect and the vehicles do not necessarily service only customers for the same cluster. This interaction of customers keeps the instance as random until $r_c > 2$ since from that point onwards the three clusters become more distinct than the fourth one around the depot and the effect of clustering gets more pronounced in the instance. Increasing r_c until 2 only makes the size of the network enclosing all the customers bigger, and therefore the cost of robust increases on these bigger random instances. When it comes to the amount of unmet demand of the deterministic, the effect of the random zone is more drastic. No matter how much the network is clustered, the unmet demand is always constant and much worse compared to set 1. The vehicles in the deterministic service customers in the random zone on their way to the clusters and usually 3 out of 4 vehicles are filled to capacity, which is not the case in the deterministic of set 1. These vehicles with full capacity are the minimum cost solutions but they have a very big potential of incurring unmet demand under uncertainty. The network structures with a scattered random demand zone around the depot as in set 2 therefore have a very negative effect on the deterministic.

When the random zone is denser around the depot, the results of set 3 are similar to the results of set 2. The deterministic results in high unmet demand values and is outperformed by the robust in almost all the cases. Having more customers in the random zone helps the robust even further. The reason why the phenomenon with 20% uncertainty disappears is due to the fact that the vehicles are close now since the number of customers they service in the random zone on their way to the clusters is significantly larger compared to set 2. Therefore when the slack in one vehicle needs to be distributed to the network, this can be achieved through customers in the random zone. The network structures with clusters and dense random zone around the depot as in set 3 therefore allow a good distribution of slack on the average and the robust solution benefits from this with little extra cost.

We conclude this experimental subsection by emphasizing that our findings in the instances by Augerat et al. (1995) are confirmed by a larger class of random instances from the population of instances with the same characteristics. In particular, we showed that both the existence of enough slack in the solution and its distribution over the network are very important factors affecting the quality of the robust. Our experiments reveal that clustered network structures with a dense random zone around the depot favor the robust. For this scenario, we showed that the deterministic could result in a large amount of unmet demand and the extra cost of the robust is relatively small.

4.4 Robust versus a uniform distribution of excess vehicle capacity

The robust solution distributes the excess vehicle capacity in the expected demand case aiming to obtain routes at minimum cost that satisfy all demand outcomes from the uncertainty set. In this section we explore how this compares to a simple strategy that uniformly distributes this excess capacity among all the vehicles.

We randomly generate instances with 4 vehicles of capacity 2100 and 68 customers with uniform expected demand of 100. Therefore there is a total of 1600 units of excess vehicle capacity to be used to address the demand uncertainty. We generate these instances according to the three sets as before. The uniform distribution of excess capacity will reserve

a buffer of 0, 100, 200, 300, and 400 units of capacity. That is, the buffer amount (excess capacity) is removed from consideration to compute the optimal deterministic solution, but it is considered when determining the amount of unmet demand that this optimal solution can face in its worst case.

To compare the quality of the solutions, we use the percent unmet demand ratio r_{ud} but generalize the percent cost ratio to $r_{rd} = \frac{z_r - z_{bc}}{z_d}$. Here we simply replaced z_d in the numerator with z_{bc} , which is the optimal objective value for the deterministic solution with reduced vehicle capacity. Figure 7 displays the average results over 30 instances for different values of percent clustering ratio r_c for Set 1 with 15% deviation in the uncertainty scenarios. Similar trends are observed in the other randomly generated sets and percent deviation values for our three types of uncertainty sets.

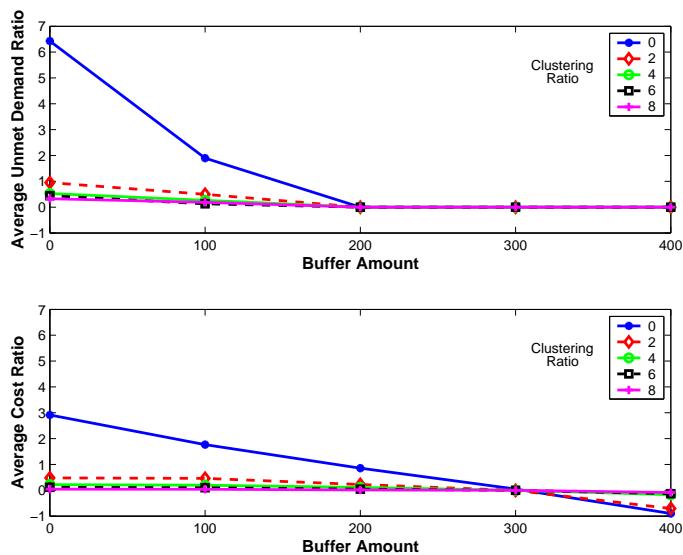


Figure 7: Comparison of Buffer Capacity and Robust solutions for Random Set 1

For a given value of percent clustering ratio, it is clear that increasing the buffer amount makes a uniform distribution of slack have less unmet demand but with an increased cost which may exceed the cost of the robust solution in some cases, giving negative values for the percent cost ratio r_{rd} . When we compare the quality of the two solutions, we see that when the buffer amount is smaller than 200, this reserve capacity is insufficient to handle

the uncertain demand. When the buffer amount is equal to 200, the uniform distribution of slack leads to a less costly solution than the robust with the same zero unmet demand. When the buffer amount is equal to 300, the two methods have the same cost with the same zero unmet demand. After this transition point (when the buffer amount is greater than 300), if we increase the buffer capacity unnecessarily, the resulting solution is more costly than the robust and is not preferable. These trends become less pronounced as the percent clustering ratio increases. That is, clustering is good for a uniform distribution of slack, which makes sense since such an even distribution of slack benefits by having each vehicle assigned to distinct, far away clusters with the same demand and uncertainty, as in the case of $r_c = 8$.

5 Conclusions

In this study, we propose to use robust optimization to obtain efficient routing solutions for problems under uncertainty. Our work has shown that robust optimization is an attractive alternative for formulating routing problems under uncertainty as it does not require distribution assumptions on the uncertainty or a cumbersome representation through scenarios.

We derived three robust counterparts for the VRP with uncertainty in demand. Our VRP formulation and definition of uncertainty sets resulted in computationally amenable RVRPs. We need to solve only a single CVRP with modified data to obtain the solution of the robust counterpart.

We used an open source VRP solver to experimentally investigate the trade-offs between a robust and deterministic solutions in terms of the increased cost of the robust solution and the possible unmet demand in the worst case of the deterministic solution. We first solved instances from the literature and postulated some insights about the network structures affecting the quality of the robust solution. We then generated random instances, with particular network structures and different degree of clustering, to validate our findings.

Our results showed that if the network structure allows a strategic distribution of the

slack in the vehicles throughout the network in such a way that the vehicles can collaborate with ease by sharing their slacks in case of uncertainty, then the robust solution is favorable on average. Such a network structure appears in a problem with clustered zones far from the depot with a dense random zone near the depot.

We also verified that the robust solution is superior than uniformly distributing among all vehicles the excess vehicle capacity under the expected demand. We showed that such a solution only competes with the robust solution if the network structure is highly clustered where the expected demand of each cluster is about the same.

References

- Augerat, P., J. Belenguer, E. Benavent, A. Corbern, D. Naddef, and G. Rinaldi (1995). Computational results with a branch and cut code for the capacitated vehicle routing problem. Research report 949-m, Universite Joseph Fourier, Grenoble, France.
- Baldacci, R., E. Hadjiconstantinou, and A. Mingozzi (2004). An exact algorithm for the capacitated vehicle routing problem based on a two-commodity network flow formulation. *Operations Research* 52(5), 723–738.
- Ben-Tal, A., A. Goryashko, E. Guslitzer, and A. Nemirovski (2003). Adjustable robust solutions of uncertain linear programs. Technical Report, Minerva Optimization Center, Technion. <http://iew3.technion.ac.il/Labs/Opt/index.php?4>.
- Ben-Tal, A. and A. Nemirovski (1998). Robust convex optimization. *Mathematics of Operations Research* 23(4), 769–805.
- Ben-Tal, A. and A. Nemirovski (1999). Robust solutions to uncertain programs. *Operations Research Letters* 25, 1–13.
- Bertsimas, D. (1988). *Probabilistic combinatorial optimization problems*. Ph. D. thesis, Operation Research Center, Massachusetts Institute of Technology, Cambridge, MA.
- Bertsimas, D. (1992). A vehicle routing problem with stochastic demand. *Operations Re-*

- search* 40(3), 574–585.
- Bertsimas, D. and M. Sim (2003). Robust discrete optimization and network flows. *Mathematical Programming* 98(1-3), 49–71.
- Bertsimas, D. and M. Sim (2004). The price of robustness. *Operations Research* 52(1), 35–53.
- Bertsimas, D. and D. Simchi-Levi (1996). A new generation of vehicle routing research: robust algorithms, addressing uncertainty. *Operations Research* 44(2), 286–304.
- Branke, J., M. Middendorf, G. Noeth, and M. M. Dessouky (2005). Waiting strategies for dynamic vehicle control. *Transportation Science* 39, 298–312.
- Charikar, M., S. Khuller, and B. Raghavachari (2001). Algorithms for capacitated vehicle routing. *SIAM J. Comput.* 31(3), 665–682.
- Daganzo, C. F. (1988). Shipment composition enhancement at a consolidation center. *Transportation Research* 22B(2), 103–124.
- Dantzig, G. B. and J. H. Ramser (1959). The truck dispatching problem. *Management Science* 6, 80–91.
- Desrochers, M. and G. Laporte (1991). Improvements and extensions to the millertuckerzemlin sub-tour elimination constraints. *Operations Research Letters* 10, 27–36.
- Dror, M. (1993). Modeling vehicle routing with uncertain demands as a stochastic program: properties of the corresponding solution. *European Journal of Operational Research* 64, 432–441.
- Dror, M., G. Laporte, and P. Trudeau (1989). Vehicle routing with stochastic demands: properties and solution framework. *Transportation Science* 23(3).
- El-Ghaoui, L., F. Oustry, and H. Lebret (1998). Robust solutions to uncertain semidefinite programs. *SIAM J. Optimization* 9(1).
- Fisher, M. (1995). Vehicle routing. In M. Ball, T. Magnanti, C. Monma, and G. Nemhauser

- (Eds.), *Network Routing*, Volume 8 of *Handbooks in Operations Research and Management Science*, pp. 1–33. Elsevier Science, Amsterdam.
- Fukasawa, R., H. Longo, J. Lysgaard, M. P. de Aragão, M. Reis, E. Uchoa, and F. R. Werneck (2006). Robust branch-and-cut-and-price for the capacitated vehicle routing problem. *Mathematical Programming* 106, 491–51.
- Gendreau, M., G. Laporte, and R. Seguin (1996). Stochastic vehicle routing. *European Journal of Operational Research* 88(1), 3–12.
- Goldfarb, D. and G. Iyengar (2003). Robust portfolio selection problems. *Mathematics of Operations Research* 28(1), 1–38.
- Jaillet, P. (1987). Stochastic routing problem. In G. Andreatta, F. Mason, and P. Serafini (Eds.), *Stochastics in Combinatorial Optimization*, pp. 178–186. World Scientific Publishing, New Jersey.
- Jaillet, P. and A. Odoni (1988). The probabilistic vehicle routing problem. In B. L. Golden and A. A. Assad (Eds.), *Vehicle Routing: Methods and Studies*. Elsevier, North-Holland, Amsterdam.
- Jézéquel, A. (1985). Probabilistic vehicle routing problems. Master’s thesis, Department of Civil Engineering, Massachusetts Institute of Technology.
- Jula, H., M. M. Dessouky, and P. Ioannou (2006). Truck route planning in non-stationary stochastic networks with time-windows at customer locations. *IEEE Transactions on Intelligent Transportation Systems* 37, 51–63.
- Lambert, V., G. Laporte, and F. Louveaux (1993). Designing collection routes through bank branches. *Computers & Operations Research* 20(7), 783–791.
- Laporte, G., F. Louveaux, and H. Mercure (1992). The vehicle routing problem with stochastic travel times. *Transportation Science* 26(3), 161–170.
- Laporte, G. and I. Osman (1995). Routing problems: a bibliography. *Annals of Operations Research* 61, 227–262.

- Lenstra, J. K. and A. H. G. Rinnooy Kan (1981). Complexity of vehicle routing and scheduling problems. *Networks* 11, 221–227.
- Lysgaard, J., N. A. Letchford, and W. R. Eglese (2004). A new branch-and-cut algorithm for the capacitated vehicle routing problem. *Mathematical Programming* 100, 423–445.
- Meyer (1996). Mohaddes associates, inc, gateway cities trucking study. Technical report, Gateway Cities Council of Governments Southeast Los Angeles County.
- Ordóñez, F., I. Sungur, and M. M. Dessouky (2005). Factors that impact solution run times of arc-based formulations of the vehicle routing problem. *USC-ISE Working paper 2005-03*.
- Palmer, K., M. M. Dessouky, and T. Abdelmaguid (2004). Impacts of management practices and advanced technologies on demand responsive transit systems. *Transportation Research, Part A: Policy and Practice* 38, 495–509.
- Ralphs, T., L. Kopman, W. Pulleyblank, and L. Trotter (2003). On the capacitated vehicle routing problem. *Mathematical Programming* 94(2-3), 343–359.
- Secomandi, N. (2001). A rollout policy for the vehicle routing problem with stochastic demands. *Operations Research* 49(5), 796–802.
- Toth, P. and D. Vigo (2002). Models, relaxations and exact approaches for the capacitated vehicle routing problem. *Discrete Applied Mathematics* 123, 487–512.
- Zhong, H., R. W. Hall, and M. M. Dessouky (2004). Territory planning and vehicle dispatching. *to appear Transportation Science*.
- Zhou, K., J. Doyle, and K. Glover (1996). *Robust and optimal control*. Prentice Hall, Upper Saddle River, NJ.