Issuing for Perishable Inventory Management with A Minimum Inventory Volume Constraint

Abstract

We study a perishable inventory system that requires to maintain a minimum inventory volume at all times, where the minimum amount in the constraint is significant with respect to regular market demand and the traditional Economic Manufacturing Quantity (EMQ) models do not suffice. The problem is motivated by applications in homeland security, which are related to the management of pharmaceuticals in the Strategic National Stockpile (SNS) for emergency preparedness. We use a modified EMQ model to represent the system and consider the issuing policy given a fixed ordering quantity, as well as the joint ordering and issuing policy problem. We aim to maximize the profit of the system under a linearly-decreasing price structure assumption. We first present the optimal issuing policy with a given ordering quantity for this modified EMQ model. Then we demonstrate that maximizing the revenue of the ordering policy with FIFO (first-in, first-out) and LIFO (last-in, first-out) issuing policies can be formulated as a non-convex non-smooth unconstrained optimization problem. The properties of both problems (e.g. optimizing issuing policy with a fixed ordering quantity and optimizing ordering policy with a fixed issuing policy) is analyzed and an efficient exact algorithm is presented to solve the joint ordering and issuing problem. We show that the LIFO issuing policy is optimal with a linearly-decreasing price structure that deteriorates with the age of the item.

Keywords: Perishable Inventory Management, Economic Manufacturing Quantity, Emergency Response, Issuing Policy

1 Introduction

In this work, we study a perishable inventory management system with a constant market demand rate in a single-manufacturer production environment and a pre-determined minimum inventory volume ($I_{min}$) must be kept in stock at all times. We assume that the price of an item is age dependent and $I_{min}$ is significantly larger than the regular market demand. This work is motivated by the supply chain management for a large-scale emergency response. An example would be the pharmaceuticals stored by the federal government and Vendor Managed Inventory (VMI) as part of the Strategic National Stockpile (SNS). For instance, the federal government keeps a stockpile of 1.2 billion Cipro pills (to be ready to treat 10 million patients in a potential anthrax attack) at any time as part of the SNS in case
of a large-scale bioterrorism attack. The shelf-life of the Cipro pills is normally 9 years and the annual ordinary demand, used as a common antibody for a number of infections, is around 300 million pills.

We assume that the primary market demand (which is exchangeable with the regular market demand in this paper) is deterministic. We also assume that there exists a secondary market (e.g., overseas market) with unlimited demand to accommodate disposals of excess inventory. In this paper, we assume that the price functions on both the primary and secondary markets are linearly decreasing in age. Customers cannot buy products in one of the two markets and sell them in the other one and the primary market price is greater than the secondary market price for an item of the same age. We assume that the system has the reliability and availability to produce for both the primary market demand and $I_{\text{min}}$ inventory.

Shen et al. (2011) proposed a modified EMQ model specially geared to a system with a high minimum volume requirement to minimize the operational cost assuming the price is not age-dependent but constant. In this paper, we use the modified EMQ model in their work and take the age-dependent price assumption into consideration. Our objective is to maximize the profit by optimizing the ordering and issuing policies while satisfying the minimum inventory requirement for this modified EMQ model. The revenue consists of selling items to the primary market and disposing excess inventory to the secondary market. Details of the cost analysis can be found in Shen (2008) and Shen et al. (2011). In this paper, we focus more on the revenue analysis with age-dependent price functions.

In the next two sections we review the relevant literature and give a brief introduction of the modified EMQ model in Shen et al. (2011). We then focus on the problems in which the price functions decrease linearly in product age. We first study the issuing policy with a fixed ordering quantity and formulate this problem into a general optimization model. From this we present the optimal issuing policy for a given ordering quantity. We then discuss the joint ordering and issuing policy problem. We describe both the cases of the FIFO and the LIFO issuing policies. For both cases, the total revenue can be represented as a non-continuous non-smooth function of the order quantity and we develop an exact algorithm to solve both cases. We also prove that the LIFO issuing policy is optimal for the joint problem. In the last two sections, we conduct numerical sensitivity experiments and discuss future research directions and provide some concluding remarks.

2 Literature Review

The ordering and issuing policies along with the disposal and pricing policies are four major researched areas in the literature of perishable inventory systems. The ordering and issuing policies have attracted the most attention. The ordering policy answers the question of when and how much to order. For perishable inventory systems, the periodic review policies are popular because they are easy to implement in practice. Fries (1975) and Nahmias (1975) performed a detailed analysis of the $m$-period model with a zero replenishment lead time. Continuous review policies have also drawn the attention of researchers. Weiss (1980) showed the $(s, S)$ policy is the optimal ordering policy for a process with a
Poisson demand and zero lead time. Liu and Lian (1999) extended Weiss (1980)’s work by considering a general renewal demand process and obtained a closed-form expected cost function. Lian and Liu (2001) further studied the continuous review problem with batch demands and allowing for backorders. Berk and Gurler (2008) showed the \((Q, r)\) policy \((r < Q)\) is a good policy in situations with Poisson demands, constant lead times, and lost sales in the presence of non-negligible ordering cost.

The issuing policy also influences the profitability because it manages the way items are removed from inventory. In many situations, the price of an item charged to a customer is dependent on its age. An appropriate issuing policy is needed to maximize the revenue. The issuing policy also affects the age distribution of the on-hand inventory. Thus the shortages and losses due to spoilage is another factor that needs to be considered when analyzing issuing policies. Derman and Klein (1958) and Lieberman (1958) were some of the earliest to discuss the optimality conditions for both the FIFO (First-In-First-Out) policy and the LIFO (Last-In-Last-Out) policy. Their discussion was in the domain where supply and demand were assumed known. Eilon (1961) and Pierskalla (1967) extended the previous work by relaxing some other restrictive assumptions (e.g., single demand source, concave or convex functions of field life, etc). Pierskalla and Roach (1972) were the first to analyze the inventory systems with random supply and demand of perishable items (a blood bank is a good example of such an inventory system). They showed that for most of the objective functions considered, the FIFO policy was the optimal policy. On the other hand, Cohen and Pekelman (1978) showed that the LIFO policy was the most suitable policy for some systems such as the military ammunition stockpile control in short conflict situations. Since the 1970s, the blood inventory management problem has attracted a significant amount of attention from the research community and motivated many studies of perishable inventory systems. Overviews of the theories and practices in blood inventory management were done by Nahmias (1982), Prastacos (1984), and Pierskalla (2004). Also motivated by problems in the blood inventory management, Goh et al. (1993) modeled the perishable inventory system with two stages. The first stage maintains the younger items and the second stage holds the older items. They considered and compared two possible two-stage FIFO policies. Other related work in the area of reserve stock include the papers by Maddah et al. (2014), Oren and Wan (1986), Zhang et al. (2009) and Zhang et al. (2009).

Most of the research in perishable inventory either studied the optimal ordering policy given an issuing policy (e.g. Ishii and Nose, 1996; Nahmias, 1982), or studied the optimal issuing policy when the order quantity is given (e.g. Pierskalla and Roach, 1972). The research in making decisions on ordering and issuing policies jointly is limited. Deniz et al. (2010) focused on how to jointly replenish and issue inventory when there are multiple streams of possibly substitutable demand for a perishable good. They provided analytical results that compare different ordering/issuing policy pairs and identified conditions that ensure one pair dominating another. Fujiwara et al. (1997) considered a two-stage perishable inventory system and derived the optimal ordering and issuing policies, where the first stage consists of the whole product made up of multiple subproducts, while the second stage consists of the subproducts. For more information on perishable inventory systems, see Goyal and Giri (2001) for recent trends and Karaesmen et al. (2011) for some future work directions.
3 Modified EMQ Model

In this section, we provide a brief summary on the modified EMQ model proposed by Shen et al. (2011), which is used to obtain the optimal ordering policy for a perishable inventory system with a large minimum volume constraint. We assume a single fixed-life perishable item is produced, consumed and stored for an infinite continuous time horizon. There are two types of demand. One is a known regular market demand with a constant rate of $D$ items per period. The production can start at any time at a constant maximum allowable rate of $P$, which is greater than $D$. The other demand is that the inventory system must maintain at every point in time at least $I_{\text{min}}$ of non-spoiled inventory. Excess supplies could be disposed to some secondary market (e.g. overseas market) as long as it is before expiration of the item. In their model, they are only concerned with the ordering policy and assumed that the inventory is consumed using the natural first-in-first-out (FIFO) policy and the prices for both the primary and secondary markets are uniform.

Fig. 1 gives an illustration of the inventory plot for the modified EMQ model. The inventory cycle $T_{\text{inv}}$ is defined as the minimum length of time that an inventory pattern repeats and it is a given parameter. It is assumed that excess supplies above the $I_{\text{min}}$ level will be disposed (salvaged) once at the end of each $T_{\text{inv}}$ so that exactly $I_{\text{min}}$ items are present at the beginning of the next inventory cycle. To maintain a fresh inventory, every $T_{\text{inv}}$, the $I_{\text{min}}$ amount produced in the previous inventory cycle is consumed and another $I_{\text{min}}$ amount is produced. The production batch size $Q$ is the sole decision variable and we can determine all the relevant quantities to describe this model as a function of $Q$. For any given $Q$, we initially run a regular EMQ cycle (underlying regular EMQ cycle with cycle length $T = \frac{Q}{D}$) and we may need to make some adjustment near the end of the inventory cycle to satisfy the minimum inventory requirement (last production cycle).

With different forms of the last production cycle and inventory cycle, there are a total of five different possible scenarios. Fig 2 provides illustrative inventory plots for all five cases. The classification is based on three criteria: 1) if a last production cycle is needed; 2) where the inventory cycle ends relative to a regular underlying EMQ cycle; 3) where the last production cycle starts. Cases 3, 4 and 5 show that an additional production cycle is needed while regular production models are sufficient in
cases 1 and 2. In cases 2 and 4, $T_{inv}$ ends in the production period of a regular underlying production cycle; otherwise in cases 1, 3, and 5, $T_{inv}$ ends in the non-production period. Furthermore, cases 3, 4 and 5 are further classified according to where the additional production cycle starts. If it starts at the same non-production period as it ends, it is case 5; if it starts at some earlier regular production cycle, we have cases 3 and 4. Detailed explanation can be found in Shen et al. (2011).

The total cost of the modified EMQ system is a non-continuous, non-differentiable function of $Q$. Let $TRC(Q)$ be the total cost function. Shen et al. (2011) show that $TRC(Q)$ for each case among the five cases is a quadratic function of the order quantity $Q$ as well as boundary conditions for transitions between the different cases. As an exemplary case, we next show $TRC(Q)$ for case 1. The total costs within a single inventory cycle is composed of 3 parts: inventory holding cost $h$, fixed setup cost $A$, and unit purchasing cost $v$. After some simplifying steps, Shen et al. (2011) shows that

\[
TRC(Q) = A(N + 1) + \left(\frac{Q}{D} - T_{inv} + N\frac{Q}{D}\right)Dv \\
+ \frac{1}{2}h[(N + 1)\frac{Q^2}{D}(1 - \frac{D}{P}) - D(\frac{Q}{D} - T_{inv} + N\frac{Q}{D})^2] \tag{1}
\]

where $N$ is the number of complete regular production cycles in one $T_{inv}$ cycle. For a given segment, $N$ is fixed so $TRC(Q)$ is a quadratic function of $Q$ for case 1. By proving that when $N$ is greater than a threshold value $\bar{N}$, the total cost will monotonically increase as $N$ increases, the exact optimal $Q$ can be obtained with pseudo-polynomial complexity. Shen et al. (2011) show that this model only holds when $I_{min} \leq T_s D$ and $T_{inv} \leq \frac{T}{2}$, where $T_s$ is the shelf life of the inventory. We also assume this condition for the remainder of this paper.

Figure 2: All Five Cases of Possible Scenarios
In this section, we are interested in maximizing the profit when both the primary and secondary market price functions decrease linearly with the age of the product. We want to determine a policy to issue products to the primary market and to dispose the remaining products to the secondary market while the ordering policy is given and fixed. Since we know when and how much to order, the cost portion of the model is a constant. Thus, we only need to consider the objective of maximizing the revenue.

An age-dependent price function is a reasonable assumption for some perishable products because customers may not be willing to pay the same price for less fresh products. Assume the shelf-life of the inventory is $T_s$. Let $f_M(t)$ be the primary market price function and $f_S(t)$ be the secondary market price function, where $t$ represents the age of an item. Throughout this paper we assume that $f_M(t)$ and $f_S(t)$ are linearly decreasing functions: $f_M(t) = a - \frac{a}{T_s} t$ and $f_S(t) = b - \frac{b}{T_s} t$. We define $a$ as the full primary market price and $b$ as the full secondary market price. Since the primary market price is decreasing by age, it is intuitive to think that if we can always issue the items with younger ages to the primary market, we might receive larger revenue. Next we prove this intuition and state the optimality condition.

We define the initial-age as the time from being produced until the beginning of the next $T_{inv}$ cycle. The possible value of the initial-age is between $[0, T_{inv}]$. Since the ordering policy is known, we have all the initial-age information of the production before every $T_{inv}$ cycle. To find the optimal issuing policy of this continuous production problem, we formulate the problem as a conceptual mixed integer programming (MIP) model. The variables are defined as follows:

- $x_{it}$: initial age of item $i$ issued at time $t$ to the primary market
- $y_j$: initial age of item $j$ disposed to the secondary market
- $z_{itk}$: binary variables representing whether $x_{it}$ equals to $k$, where $z_{itk} = \begin{cases} 1 & \text{if } x_{it} = k \\ 0 & \text{if } x_{it} \neq k \end{cases}$
- $w_{jk}$: binary variables representing whether $y_j$ equals to $k$, where $w_{jk} = \begin{cases} 1 & \text{if } y_j = k \\ 0 & \text{if } y_j \neq k \end{cases}$

Please note that we state this problem as a conceptual mixed integer programming problem because this model contains an infinite number of binary decision variables ($z_{itk}$ and $w_{jk}$) and infinite number of constraints, as the binary decision variables are defined over a continuous feasible range. This makes the $\forall i, j, t, k$ notation invalid strictly speaking as they cannot be enumerated as in discrete domains. However, we still use such notation to demonstrate the conceptual idea of this model and we do not intend to solve this model analytically.

The objective is to maximize the revenue from all sales. The objective function can be written as

\[
\int_0^{T_{inv}} \int_0^{D} \left[ a - \frac{a}{T_s} (x_{it} + t) \right] di \, dt + \int_0^{(P-D)T_{inv}} \left[ b - \frac{b}{T_s} (y_j + T_{inv}) \right] dj
\]
Rearranging the objective function and removing the constant terms. We have the MIP model:

\[
\begin{align*}
\min & \quad a \int_0^{T_{inv}-1} \int_0^D x_{it} \, di \, dt + b \int_0^{(P-D)T_{inv}} y_j \, dj \\
\text{s.t.} & \quad \int_0^{T_{inv}} \int_0^D z_{itk} \, di \, dt + \int_0^{(P-D)T_{inv}} w_{jk} \, dj = PT_{inv} \quad \forall k \\
& \quad \int_0^{T_{inv}} z_{itk} \, dk = 1 \quad \forall i, t \\
& \quad \int_0^{T_{inv}} w_{jk} \, dk = 1 \quad \forall j \\
& \quad \int_0^{T_{inv}} k \cdot z_{itk} \, dk = x_{it} \quad \forall i, t \\
& \quad \int_0^{T_{inv}} k \cdot w_{jk} \, dk = y_j \quad \forall j \\
& \quad x_{it} \in [0, T_{inv}], \quad y_j \in [0, T_{inv}], \quad z_{itk} \in \{0,1\}, \quad w_{jk} \in \{0,1\} \\
& \quad i \in [0, D], \quad j \in [0, (P-D)T_{inv}], \quad t \in [0, T_{inv}], \quad k \in [0, T_{inv}] \\
\end{align*}
\]

Constraint (2) guarantees that there are exactly \( P \) items with initial age \( k \) issued to either the primary or the secondary market over one \( T_{inv} \) cycle. Constraints (3) and (4) state that each item issued to either the primary or the secondary market has a unique initial age. The relationships between the binary variables and integer variables are established in constraints (5) and (6).

**Theorem 1** If \( a \geq b \), it is optimal to issue the youngest items to the primary market demand when we have a linearly decreasing price function.

**Proof.** See Appendix A for proof. ■

5 Joint Ordering and Issuing Policy Problem

In the previous section, we only maximize the revenue since with a given \( Q \) the cost is fixed. Now we aim to maximize the profit by optimizing the ordering and issuing policies jointly. Below is a list of notation that we use in our model.

- \( D \): primary market demand rate
- \( P \): production rate
- \( T_{inv} \): inventory cycle length that an inventory pattern repeats
- \( T_s \): shelf-life of the inventory
- \( T \): regular production cycle length
- \( a \): full primary market price
- \( b \): full secondary market price
- \( N \): number of complete regular production cycles in one \( T_{inv} \) cycle
5.1 FIFO Issuing

A detailed process to develop the revenue functions for case 1 is provided. Similar logic follows for the other cases and the details are shown in Appendix B. The revenue function can be decomposed into revenue from the primary market and revenue from the secondary market. First the revenue from the primary market for a given \( Q \) in case 1 is developed (see Fig. 3). Assume with FIFO, the product issued at time \( t \) (where \( t \in [0, T_{inv}] \)) is produced at time \( t' \); thus the age of the product issued at time \( t \) is \( T_{inv} + t - t' \). We note that \( t' \) can only occur in a production period. We can also see that \( t'P = tD \) since the products we consumed in the current cycle are produced from the previous cycle in a FIFO manner. Hence the age of the product issued at time \( t \) is \( T_{inv} + t - tD_{P} \), which is a function of \( t \). For a regular production cycle, we can derive the revenue function of satisfying the primary market demand from time 0 to time \( T \) as

\[
\int_{0}^{T} [a - (T_{inv} + t - \frac{tD}{P}) \frac{a}{T_{s}}] D \, dt. \tag{7}
\]

The revenue of satisfying the demand from time \( T \) to time \( 2T \) would be the same as (7) and can be generalized to the case where there are \( N \) complete production cycles in \( T_{inv} \). The same idea is also used to find the revenue from time \( NT \) to \( T_{inv} \) and it would be the same as that from 0 to \( T_{inv} - NT \). Thus, the revenue of satisfying the primary market demand in the \( T_{inv} \) cycle is

\[
N \int_{0}^{T} [a - (T_{inv} + t - \frac{tD}{P}) \frac{a}{T_{s}}] D \, dt + \int_{0}^{T_{inv} - NT} [a - (T_{inv} + t - \frac{tD}{P}) \frac{a}{T_{s}}] D \, dt. \tag{8}
\]

Next, we derive the revenue for the items disposed to the secondary market for a given \( Q \). Note that we assume an unlimited demand for the items disposed to the secondary market. In this case, the amount of disposal \( I_{dis} = ((N+1)T - T_{inv})D \). Assume the units which are to be disposed are produced

Figure 3: Graph illustration for Case 1 under FIFO
in the time period from \( t_1 \) to \( t_2 \), where \( t_2 \) is the last production time in the previous cycle. Let \( T_1 \) be the production time in one regular production cycle and we have \( t_2 - t_1 = T_1 - (T_{inv} - NT) \frac{P}{T_s} \). Let \( s \) be the time from \( t_1 \) to the end of the \( T_{inv} \) cycle. We have \( s = T_{inv} - NT - T_1 + (t_2 - t_1) \). If the problem is a discrete production one, the first disposed unit would have age \( T_{inv} + s = T_{inv} + (T_{inv} - NT - T_1 + t_2 - t_1 - \frac{1}{T}) \). The \( t \)th unit disposed would have age \( T_{inv} + (T_{inv} - NT - T_1 + t_2 - t_1 - \frac{1}{T}) \) and the last unit disposed would have age \( T_{inv} + (T_{inv} - NT - T_1) \). Without loss of generality, we can correspond the age results of the discrete production problem to the results of the continuous one. Thus the revenue from salvaging the excess items is

\[
\int_0^{t_{dis}} [b - (2T_{inv} - NT - T_1 + \frac{I_{dis}}{P} - \frac{x}{P}\frac{b}{T_s})] dx. \tag{9}
\]

Combining the revenue from the primary and secondary markets (equations (8) and (9)), we have

\[
N \int_0^T [a - (T_{inv} + t - t\frac{D}{P}) \frac{a}{T_s}]D dt + \int_0^{T_{inv} - NT} [a - (T_{inv} + t - t\frac{D}{P}) \frac{a}{T_s}]D dt + \int_0^{t_{dis}} [b - (2T_{inv} - NT - T_1 + \frac{I_{dis}}{P} - \frac{x}{P}\frac{b}{T_s})] dx \tag{10}
\]

where \( I_{dis} = ((N + 1)T - T_{inv})D \). Note that for a regular production cycle, the cycle length \( T = \frac{Q}{P} \) and the production time \( T_1 = \frac{D}{P}T \). With further simplification, the total revenue (equation (10)) as a function of \( Q \) can be expressed as

\[
\text{Revenue}(Q) = x \cdot Q^2 + y \cdot Q + z,
\]

where

\[
x = \frac{b(N + 1)^2}{2PT_s} - \frac{(1 - \frac{D}{P})aN^2}{2T_sD} - \frac{N(1 - \frac{D}{P})a}{2T_sD} - \frac{(-\frac{N}{T} - \frac{1}{P} + \frac{N+1}{TP})b(N + 1)}{T_s},
\]

\[
y = -\left[\frac{(2T_{inv} - \frac{T_{inv}D}{P})b(N + 1)}{T_sD} - \frac{(-\frac{N}{T} - \frac{1}{P} + \frac{N+1}{TP})bT_{inv}}{T_s}\right]D + \frac{(1 - \frac{D}{P})aT_{inv}N}{T_s} + b(N + 1) + \frac{bT_{inv}(N + 1)D}{PT_s}
\]

\[
z = \frac{(2T_{inv} - \frac{T_{inv}D}{P})bT_{inv}D}{T_s} + \left(a - \frac{T_{inv}a}{T_s}\right)DT_{inv} - \frac{(1 - \frac{D}{P})aDT_{inv}^2}{2T_s} + \frac{bT_{inv}^2D^2}{2PT_s} - bT_{inv}D.
\]

### 5.2 LIFO Issuing

The revenue functions have to be rewritten for each case under the LIFO policy. Again, a detailed process to develop the revenue functions for case 1 is provided and the details for the other cases are shown in Appendix B.

Assume under the LIFO policy the item issued at time \( s_1 \) is produced at time \( s_1' \) (see Fig. 4); thus the age of this product is \( t_1 + t_1' + (T_{inv} - 2T - T_1) \), where \( T_1 \) denotes the production time in one regular production cycle. And we can also see that \( t_1D = t_1'P \) since the products we consumed in the current cycle are produced from the previous cycle in a LIFO manner. Thus the revenue for satisfying
the primary market demand in time period $[0, T]$ is

$$\int_0^T [a - (t_1 + \frac{t_D}{P} + T_{inv} - 2T - T_1) \frac{a}{T_s} ] D dt_1.$$ 

Now assume the products issued at $s_2$ are produced at time $s'_2$ and have the age of $t_2 + t'_2 + (T_{inv} - 2T - T_1) + 2T$, where $t_2D = t'_2P$. The revenue from satisfying the primary market demand in time period $[T, 2T]$ is

$$\int_0^T [a - (t_2 + \frac{t_2D}{P} + T_{inv} - T_1) \frac{a}{T_s} ] D dt_2.$$ 

Generalizing this to the case where there are $N$ complete production cycles in $[0, T_{inv}]$, we have the revenue for period $[0, NT]$ as below

$$\sum_{n=0}^{N-1} \left\{ \int_0^T [a - (t + \frac{tD}{P} + T_{inv} - NT - T_1 + 2nT) \frac{a}{T_s} ] D dt \right\}. \quad (11)$$ 

Similarly, for the time period $[2T, T_{inv}]$ in this case, the revenue is

$$\int_0^{T_{inv}-2T} [a - (t + \frac{tD}{P} + T_{inv} - T_1 + 2T) \frac{a}{T_s} ] D dt$$

which can be generalized to

$$\int_0^{T_{inv}-NT} [a - (t + \frac{tD}{P} + T_{inv} - T_1 + NT) \frac{a}{T_s} ] D dt. \quad (12)$$

Next, we derive the revenue for the items disposed to the secondary market for a given $Q$. In this case, the amount of disposal $I_{dis} = ((N + 1)T - T_{inv})D$. Assume the first item of disposal will be produced at time $s_3$ and the last item of disposal will be produced at $s'_3$. We can see that $I_{dis} = t_3P$. The $i^{th}$ item of disposal will have age $T_{inv} + T_{inv} - t_3 + \frac{(i-1)}{P}$. Thus the revenue from disposal can be written as

$$\int_0^{I_{dis}} [b - (2T_{inv} - \frac{x}{P} \frac{b}{T_s} ] dx. \quad (13)$$
Now combining the revenue from both markets (equations (11), (12), and (13)), we have

\[
\sum_{n=0}^{N-1} \left\{ \int_0^T \left[ a - (t + \frac{tD}{P}) + T_{\text{inv}} - NT - T_1 + 2nT \frac{a}{T_s} \right] D \, dt \right\} \\
+ \int_0^{T_{\text{inv}}-NT} \left[ a - (t + \frac{tD}{P}) + T_{\text{inv}} - T_1 + NT \frac{a}{T_s} \right] D \, dt \\
+ \int_0^{I_{\text{dis}}} \left[ b - (2T_{\text{inv}} - \frac{x}{P}) \frac{b}{T_s} \right] dx
\]

where \( I_{\text{dis}} = ((N + 1)T - T_{\text{inv}})D \). With further simplification, the total revenue (equation (14)) as a function of \( Q \) can be expressed as Revenue(\( Q \)) = \( x \cdot Q^2 + y \cdot Q + z \), where

\[
x = \frac{(2N - \frac{1}{P}) aN T_s}{2T_s} - \frac{(1 + \frac{P}{D}) aN^2}{2T_s D} + \frac{aN}{2DT_s} \left( a - \frac{aT_{\text{inv}}}{T_s} \right) + \frac{b(1 + N)^2}{2PT_s}
\]

\[
y = b(1 + N) + \frac{(1 + \frac{P}{D}) aT_{\text{inv}} N}{T_s} - \frac{NaT_{\text{inv}}}{T_s} - \left( a - \frac{aT_{\text{inv}}}{T_s} \right) N - \frac{(2N - \frac{1}{P}) aDT_{\text{inv}}}{T_s} - \frac{bT_{\text{inv}}(1 + N)D}{PT_s} - \frac{2T_{\text{inv}}b(1 + N)}{T_s} + Na
\]

\[
z = -T_{\text{inv}}bD + \frac{2T_{\text{inv}}^2 2bD}{T_s} + \left( a - \frac{aT_{\text{inv}}}{T_s} \right) DT_{\text{inv}} - \frac{(1 + \frac{P}{D}) aDT_{\text{inv}}^2}{2T_s} + \frac{bT_{\text{inv}}^2 D^2}{2PT_s}
\]

5.3 Exact Algorithm

With the total revenue calculation formula in the previous subsections and the total cost calculation in Shen et al. (2011), we can obtain the numerical total profit as a function of \( Q \). The total profit is non-continuous and non-differentiable for both the FIFO and LIFO policies. In this subsection, we propose an exact solution method to obtain the optimal \( Q \) that reaches the maximum profit with pseudo-polynomial complexity.

**Theorem 2 (Local Optimality)** Within each case with fixed \( N \) and \( M \), the total profit is a quadratic function of \( Q \) and the local maximum is either at a boundary point or at the zero first order derivative point for both the FIFO and LIFO issuing policies.

**Proof.** From the previous subsections, we know that the total revenue with fixed \( N \) and \( M \) for each case is a quadratic function of \( Q \). The total cost with fixed \( N \) and \( M \) for each case is also shown as a quadratic function to \( Q \) in Shen et al. (2011). The proof is immediate. ■

Next, we show that we only need to compute and compare a limited number of local optimal values to find the global optimum. In Shen et al. (2011), they demonstrated that the global optimum to minimize the total cost is located near the region where the ordering cost and the inventory carrying cost are most balanced (as in the regular EMQ model). They have shown that cases 1, 2, and 5 may...
only occur when the $N$ value is small and proved that the total cost would monotonically increase as $N$ increases after a threshold value $\bar{N}$ for cases 3 and 4. We use a similar approach to prove the property which guarantees the global optimality to maximize the total profit.

**Theorem 3 (Global Optimality)** When $N$ is greater than a threshold value $\bar{N}$, the total profit will monotonically decrease as $N$ increases.

**Proof.** See Appendix A for proof. ■

Guaranteed by Theorem 2 and Theorem 3, below we have the algorithm to reach the global optimum for the joint ordering and issuing policy with FIFO and LIFO on the modified EMQ model with pseudo-polynomial complexity.

**Initialization:**
1. Calculate $\hat{N}$
2. Set global maximum $g_{\text{max}} = -\infty$

**Main Loop:** for each $N$ from 0 to $\hat{N}$
1. Calculate possible cases (at most five cases) and their boundary points (for FIFO or LIFO policies)
2. Record local optimum for each case $i$ ($= l_{maxN}[i]$)
3. Get the local optimum for current $N$ among all possible cases ($l_{\text{maxN}} = \max_i l_{maxN}[i]$)
4. If $l_{\text{maxN}} > g_{\text{max}}$, then set $g_{\text{max}} = l_{\text{maxN}}$

Next, we present the threshold value $\hat{N}$ and prove that the above algorithm has polynomial complexity. Let $T_a$ and $T_b$ represent the total production time and idle time for the regular EMQ cycle in a $T_{inv}$ cycle respectively. Let $A$ be the fixed ordering cost. $T_{triangle}$ denote the time that is occupied by the regular EMQ cycle triangles.

**Theorem 4 (Complexity)** The exact solution algorithm at most needs to explore $5\hat{N}$ local maximum points to reach the global maximum, where $\hat{N} = \lfloor \sqrt{\frac{D^2(1 - \frac{D}{P})T_{triangle}^2 + b\frac{T_aT_b}{2}}{1 + \frac{1}{A}}} \rfloor$.

**Proof.** From the proof in Theorem 3, we have $\delta = \left\lfloor \frac{b\frac{T_aT_b}{2} + \frac{D}{2}(1 - \frac{D}{P})T_{triangle}^2}{N(N+1)} - A \right\rfloor$, and the monotonically increasing threshold is given by $\hat{N}$ where $\hat{N}$ is the smallest integer value satisfied by $\delta \leq 0$. Hence the smallest $N$ which satisfies the above inequality is $\hat{N} = \left\lfloor \sqrt{\frac{D^2(1 - \frac{D}{P})T_{triangle}^2 + b\frac{T_aT_b}{2}}{1 + \frac{1}{A}}} \right\rfloor$. For each $N$, there are at most five different cases, thus the maximum number of local maximum points we need to explore before reaching the global optimality is $5\hat{N}$. ■
5.4 The Optimal Joint Ordering and Issuing Policy

By Theorem 1, it implies that, for any order quantity \(Q\), the revenue of the LIFO policy is larger than that of the FIFO policy. Since the cost is not a function of the issuing policy, we have

\[
\text{Profit}_{\text{LIFO}}(Q) \geq \text{Profit}_{\text{FIFO}}(Q) \tag{15}
\]

where the profit functions can be obtained by subtracting the cost functions from the revenue function. For example, for case 1:

\[
\text{Profit}_{\text{FIFO}}(Q) = \text{Revenue}_{\text{FIFO}}(Q) - \text{TRC}(Q) = \text{equation}(10) - \text{equation}(1)
\]

\[
\text{Profit}_{\text{LIFO}}(Q) = \text{Revenue}_{\text{LIFO}}(Q) - \text{TRC}(Q) = \text{equation}(14) - \text{equation}(1)
\]

Let \(Q^*_{\text{FIFO}}\) and \(Q^*_{\text{LIFO}}\) denote the optimal order quantities that generate the maximal profit under the FIFO and LIFO policies, respectively. That is, \(Q^*_{\text{FIFO}}\) is the maximizing argument of the function \(\text{Profit}_{\text{FIFO}}(Q)\) and \(Q^*_{\text{LIFO}}\) is the maximizing argument of the function \(\text{Profit}_{\text{LIFO}}(Q)\).

By substituting \(Q\) with \(Q^*_{\text{FIFO}}\) in equation (15), we have

\[
\text{Profit}_{\text{LIFO}}(Q^*_{\text{FIFO}}) \geq \text{Profit}_{\text{FIFO}}(Q^*_{\text{FIFO}}) \tag{16}
\]

Since \(Q^*_{\text{LIFO}}\) is the optimal order quantity under the LIFO policy, we have

\[
\text{Profit}_{\text{LIFO}}(Q^*_{\text{LIFO}}) \geq \text{Profit}_{\text{LIFO}}(Q^*_{\text{FIFO}}) \tag{17}
\]

By equation (16) and (17), we know that

\[
\text{Profit}_{\text{LIFO}}(Q^*_{\text{LIFO}}) \geq \text{Profit}_{\text{FIFO}}(Q^*_{\text{FIFO}}) \tag{18}
\]

Hence, we can conclude that the maximal profit of the LIFO policy is no less than that of the FIFO policy. Next, we show the optimal production batch size and the issuing policy for the joint problem.

Theorem 5 \((Q^*_{\text{LIFO}}, \text{LIFO})\) is an optimal policy for the joint modified EMQ problem.

Proof. Assume there is an optimal issuing policy \(A\) with optimal ordering quantity \(Q\), called \((Q, \text{Policy}_A)\), for the joint problem. By Theorem 1, for the same \(Q\) and varying the issuing policy, \((Q, \text{LIFO})\) is also the optimal policy. By optimality, for the fixed issuing policy LIFO, \((Q^*_{\text{LIFO}}, \text{LIFO})\) is also an optimal policy. ■

Theorem 5 can then be re-written as \(\text{Profit}_{\text{LIFO}}(Q^*_{\text{LIFO}}) \geq \text{Profit}_{\text{AnyPolicy}}(Q^*_{\text{AnyPolicy}})\).
6 Computational Experiments

In the previous section, we have shown the solution approaches for the joint ordering and issuing problem and proven that \((Q^*_LIFO, LIFO)\) is the optimal solution. We are also interested in how much the LIFO policy dominates over the FIFO policy under different scenarios. In this section, we conduct numerical experiments on data for Cipro pills to compare the maximum average return per item under the FIFO and LIFO policies for a linearly-decreasing price model. The maximum average return per item is the maximum total profit in a \(T_{inv}\) cycle divided by the \(I_{min}\) amount.

Our experiments use some of the parameter settings as Shen et al. (2011). In a potential anthrax attack, the U.S. federal government is prepared to build up a stockpile of 1.2 billion Cipro pills. Thus, we use 1.2 billion for \(I_{min}\). We assume that Cipro has a shelf-life of 9 years \(T_s\) and it has to be discarded when it expires. The pills can be sold as long as they are within the shelf-life. Since \(T_{inv}\) can be at most half the shelf-life and we want to leave some flexibility for the manufacturers to refresh their inventory in a shorter period, we use 36 months for \(T_{inv}\) as the base case. We choose the prices for the base case according to a survey from the Health Action International Foundation (2009). We use the price of the originator brand product (manufactured by Bayer) in the U.S. of $4.5 as the full primary market price per pill \((a)\). The secondary market could be any overseas market and the sale price may vary based on location (e.g., $1.25 per pill in Southern East Asia or $3.12 per pill in Europe). For our base case, we use $2 as the full secondary market price per pill \((b)\). We assume that the market prices decrease linearly as an item ages. According to Singh (2001), Bayer sold $1.04 billion worth of Cipro in 1999, which is equivalent to about 220 million pills in 1999; with a 4\% per year growth rate, the demand reaches about 340 million in 2010/2011 and we use this as the demand rate \(D\). The rest of the parameter settings for the base case are shown in Table 1.

The maximum possible \(Q\) is the maximum production capacity in an inventory cycle, \(PT_{inv}\). In this extreme case, the system produces continuously (case 2) and incurs the maximum disposal amount for the secondary market. However, we aim to satisfy the demand for the primary market and \(I_{min}\)

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Unit</th>
<th>Estimation</th>
</tr>
</thead>
<tbody>
<tr>
<td>(I_{min})</td>
<td>million pills</td>
<td>1200</td>
</tr>
<tr>
<td>(T_s)</td>
<td>month</td>
<td>108</td>
</tr>
<tr>
<td>(T_{inv})</td>
<td>month</td>
<td>36</td>
</tr>
<tr>
<td>(a)</td>
<td>$ /pill</td>
<td>4.5</td>
</tr>
<tr>
<td>(b)</td>
<td>$ /pill</td>
<td>2</td>
</tr>
<tr>
<td>(D)</td>
<td>mil pill / year</td>
<td>340</td>
</tr>
<tr>
<td>(P)</td>
<td>mil pill / year</td>
<td>600</td>
</tr>
<tr>
<td>(v)</td>
<td>mil $ /mil pill</td>
<td>0.2</td>
</tr>
<tr>
<td>(A)</td>
<td>mil $ /time</td>
<td>2</td>
</tr>
<tr>
<td>(h)</td>
<td>mil $ /mil pill/year</td>
<td>0.02</td>
</tr>
</tbody>
</table>
and we do not produce for the secondary market. Thus, we limit the maximum production batch size to be $0.75PT_{\text{inv}}$ in order to avoid our system from constantly producing for disposal.

We first fix the full primary market price $a = 4.5$ and use the full secondary market price $b$ as a changing variable. Fig. 5(a) illustrates the maximum average return per item under the FIFO and LIFO policies for various $b$ values. The horizontal axis represents the full secondary market price $b$ ranging from $0.1$ to $3$ and the vertical axis represents the maximum average return per item in dollars. Note that the maximum average return is increasing in the full secondary market price $b$ under any fixed issuing policy. Thus, as can be seen from the figure, both the FIFO and LIFO policies increase in the maximum average return as $b$ gets larger. It is intuitive to think that the maximum average return is proportional to $b$. However, since $Q^*_{\text{FIFO}}$ and $Q^*_{\text{LIFO}}$ may change for different $b$ values which affects the revenue and cost values, the maximum average return is an affine function instead of a linear function. The circle markers in Fig. 5(a) represent those prices $b$ when the optimal order quantity changes. Note that for the LIFO policy the optimal $Q^*_{\text{LIFO}}$ stays at 180 for all values of $b$ in this experiment.

Now we fix the full secondary market price $b$ at $2$ and compare the maximum average return per item under the FIFO and LIFO policies in terms of different full primary market price $a$. Fig. 5(b) shows the maximum average return per item for both policies. The horizontal axis represents the full primary market price $a$ ranging from $3$ to $5$ and the vertical axis represents the maximum average return per item in dollars. Note that the maximum average return is also increasing in $a$ under any fixed issuing policy. The circle markers in Fig. 5(b) represent those prices $a$ when the optimal order quantity changes. We can see that the plot lines for the FIFO and LIFO policies look like straight lines, even when the optimal order quantity changes several times in FIFO. Generally speaking, the revenue from the primary market sale dominates the total profit. Hence, even though the optimal order quantities and the cost may vary at different prices $a$, the maximum profit is almost proportional to $a$ under both policies and thus so is the maximum average return per item.

In Fig. 5(a), as $b$ gets larger, the difference between the FIFO and LIFO policies increases ranging

![Figure 5](attachment:image.png)

Figure 5: Maximum average return per item under the FIFO and LIFO policies for different $a$ and $b$ values
Figure 6: Difference in maximum average return per item between the FIFO and LIFO policies for different $D$ for different $a$ and $b$ values

from around $0.20$ to $0.23$. In Fig. 5(b), as $a$ gets larger, the difference also increases ranging from around $0.17$ to $0.27$. We are interested in studying how the gap in the maximum average return between the FIFO and LIFO policies changes under different scenarios at different price levels, $I_{min}$, and $D$ levels. Unless otherwise specified, we use the base case settings in Table 1.

First, we examine the impact of price changes at different $D$ levels. Fig. 6(a) illustrates the differences in the maximum average return per item between the FIFO and LIFO policies in terms of various $b$. Different plot lines denote different $D$ levels. As can be seen from the figure, when $D \leq 330$, the gap between the maximum average return of the two policies increases in $b$. That is, the maximum average return of the LIFO policy increases faster than that of the FIFO policy as $b$ gets larger. On the other hand, when $D$ is larger than 330, the growth rate in the maximum average return of the LIFO policy is not necessarily larger than that of the FIFO policy. Thus, the smaller the $D$, the faster the maximum average return of the LIFO policy increases than that of the FIFO policy as $b$ increases.

Fig. 6(b) illustrates the differences in the maximum average return per item between the FIFO and LIFO policies in terms of various $a$ at different $D$ levels. Since the maximum average return is almost proportional to $a$, as we mentioned before, the plot lines look like straight lines. We also observe that the gap increases in $a$, that is, the maximum average return under the LIFO policy increases faster than that under the FIFO policy as $a$ gets larger. From Fig. 6(a) and Fig. 6(b), we can conclude that when $D$ is smaller, the LIFO policy tends to outperform the FIFO policy much more.

Fig. 7(a) and Fig. 7(b) show the plots of the gap of the maximum average return per item between the FIFO and LIFO policies at different $I_{min}$ levels in terms of various $b$ and $a$ values, respectively. Different plot lines denote different $I_{min}$ levels. In Fig. 7(a), when $I_{min} > 1200$, the maximum average return of the LIFO policy increases faster than that of the FIFO policy as $b$ increases. When $I_{min} \leq 1200$, the growth rate of the LIFO policy is not necessarily larger than that of the FIFO policy. Thus, the larger the $I_{min}$, the faster the maximum average return of the LIFO policy increases than that of the FIFO policy as $b$ increases. In Fig. 7(b), we observe that the gap increases in $a$ and the plot lines look like straight lines. When $I_{min}$ is larger, the gap between the maximum average return
of the FIFO and LIFO policies is larger. From Fig. 7(a) and Fig. 7(b), we may conclude that when \( I_{\text{min}} \) is larger, the LIFO policy tends to dominate the FIFO policy more.

7 Conclusion and Future Research

In this paper, we study a perishable inventory management system with a minimum inventory quantity constraint in response to large-scale emergencies. To maximize the profit by optimizing the ordering and issuing policies, we consider a linearly-decreasing price function model based on the modified EMQ model in Shen et al. (2011). First, we study the issuing policy with a given order quantity; optimal issuing policies are derived for the given order quantity. Next, we derive the revenue functions when the FIFO or LIFO issuing policies are used. We prove the existence of the global maximum of profit for the joint problem with FIFO and LIFO issuing policies. Hence, an exact solution procedure with pseudo-polynomial complexity is proposed. We also prove that the LIFO issuing policy and the joint problem’s optimal solution is \( Q_{\text{LIFO}}^* \) for the linear price function.

The basic assumption of the paper is that the emergency events are rare and that it is more likely that the items will reach their expiration date before an emergency occurs. Therefore in this research we did not investigate the impact of the emergency on the inventory. Future research can extend the model to consider the implication of the occurrence of emergency events on the order quantity and cycle. If the emergency demand is greater than the inventory then the current inventory is depleted immediately and we need plans to meet the unsatisfied demand and replenish the stockpile; if the emergency demand is less than the current inventory, we need plans to replenish the stockpile to the minimum inventory level. Another extension is to take the pricing policy into consideration. Our current model assumes the price function is given and we determine the ordering and issuing policies based on the price structure. Thus, we might be interested in deciding the pricing, ordering, and issuing strategies simultaneously. Furthermore, we now assume the problem with deterministic primary market demand. It is also of interest to study the problem under stochastic demand.
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Appendices

A Proof of Theorem 1 and Theorem 3

**Theorem 1**  
If \( a \geq b \), it is optimal to issue the youngest items to the primary market demand when we have a linearly decreasing price function.

**Proof.** We first consider a problem that has a continuous production plan with disposal, i.e., without the downhill no-production periods. Let \( \hat{X} \) be the set of initial-age of the items issued to the primary market by using the issuing policy that assigns the youngest items to the primary market and \( \hat{Y} \) be the set of initial-age of the remaining items disposed to the secondary market. Similarly, let \( X' \) and \( Y' \) be the sets of initial-age by using any other issuing policy for the primary and secondary markets, respectively. Note that \( \hat{X} \cup \hat{Y} = X' \cup Y' = S \), where \( S \) is the set of initial-age of the items produced from the previous \( T_{inv} \) cycle, i.e., \( S = [0, T_{inv}] \) and for each age \( t \in [0, T_{inv}] \), there are \( P \) items in \( S \). Assume \( \int \int \hat{x}_{it} \, dt = m \) and \( \int \int x'_{it} \, dt = n \). We can see that \( m \leq n \). Since \( \hat{X} \cup \hat{Y} = S \) and \( X' \cup Y' = S \), let \( \hat{c} \) be the integral of all the elements in \( S \) and we have \( \int \int \hat{x}_{it} \, dt + \int \hat{y}_{j} \, dj = \hat{c} \) and \( \int \int x'_{it} \, dt + \int y'_{j} \, dj = \hat{c} \). That is,

\[
a \int \int \hat{x}_{it} \, dt + b \int \hat{y}_{j} \, dj = ma + (c - m)b \tag{19}
\]

\[
a \int \int x'_{it} \, dt + b \int y'_{j} \, dj = na + (c - n)b \tag{20}
\]

Because \( m \leq n \), we have \( (20) - (19) = (n - m)(a - b) \geq 0 \) if \( a \geq b \). Thus if \( a \geq b \), we can guarantee that by issuing the youngest items to the primary market demand, we will obtain the optimal value for the continuous production problem.

Now consider the non-continuous production problem (as in Fig. 1). In a \( T_{inv} \) cycle, there will be some time segments without production. The set \( S \) now no longer contains every value between \( [0, T_{inv}] \), some discontinuity in age exists. Except for the difference in the parameter \( S \), the rest of the previous MIP model can be used for the non-continuous production problem and we can then claim the above proof is still valid.

**Theorem 3 (Global Optimality)**  
When \( N \) is greater than a threshold value \( \hat{N} \), the total profit will monotonically decrease as \( N \) increases.

**Proof.** We first prove that the total revenue of a fixed \( N \) value for both cases 3 and 4 is bounded; then we prove that the bound monotonically increases as \( N \) increases after a certain threshold value \( N' \). In Shen et al. (2011), they proved that the cost of this system increases monotonically after a threshold
(N̂) as well. We further prove that the cost increase dominates the revenue increase after threshold (N̂), hence the total profit decreases in N after N > N̂.

Figure 8: Graph Illustration for the Global Optimum Proof (part I)

Figure 9: Graph Illustration for the Global Optimum Proof (part II)
In this proof, we use the total age of the inventory to represent the increasing or decreasing trends of total revenue since the price function is linearly decreasing in age. Fig. 8 depicts the inventory plot and the age distribution of the inventory at the beginning of the parameter $T_{inv}$: 8(a1), 8(a2), and 8(a3) give the inventory plot (the horizontal axis is time and the vertical axis is the inventory level above $I_{min}$) for $N = 1, 2, 3$; 8(b1), 8(b2), and 8(b3) plot the corresponding age distribution of the inventory (the horizontal axis is time and the vertical axis is age). $T_a$ and $T_b$ represent the total production time and idle time for the regular EMQ cycle in a $T_{inv}$ cycle respectively; and $T_c$ represents the length of the last production cycle. The total area of the tilted line shadow region represents the total age of the complete inventory at the beginning of $T_{inv}$. The total production time is always ($T_a + T_c$) regardless the value of $N$, which implies that the total area of both the tilted line shadow region and horizontal line shadow region are constant for all values of $N$. Therefore we can first compute the area of the horizontal line shadow region and derive the total age of inventory from it. We observe that the last $T_c$ region area is fixed for all values of $N$. We can further simplify the calculation on the horizontal line shadow area of the first $N$ slots from the left (as the $N + 1$ region is for $T_c$ and fixed).

We use the graphs in Fig. 8 and Fig. 9 to prove the bounds on the total revenue with a fixed $N$. Fig. 9(a2) illustrates a general situation of the inventory plot with three complete and one incomplete regular EMQ orderings before the last production cycle. Its inventory age distribution plot can be represented in Fig. 9(b2). As explained earlier, the area that we need to compute has been highlighted with grid-lines. The grid-line shadow area in Fig. 9(b2) is bounded by the grid-line shadow area of the three slender slots in Fig. 9(b1) (which corresponds to three smaller triangles in Fig. 9(a1) inventory plot) from below and is bounded by the grid-line shadow area of the four wider slots in Fig. 9(b3) (which corresponds to four larger triangles in Fig. 9(a3) inventory plot) from above. This is evident since the last production cycle triangle is the same for all three situations (Fig. 9(a1), 9(a2), and 9(a3)) and the time $T_{inv} - T_c$ is evenly divided by 3 and 4 respectively in Fig. 9(a3) and Fig. 9(a1).

Thus the width of one slot in Fig. 9(b2) $w$ is smaller than the slot width $T_a/3$ in Fig. 9(b3) and is larger than the slot width $T_a/4$ in Fig. 9(b1). Since the straight line separates the tilted line shadow region and grid line shadow region is $f(t) = t$, the grid line shadow area in Fig. 9(b2) is between the grid line shadow area in Fig. 9(b1) and Fig. 9(b3). We use $Area_{3+}$ to denote the grid line shadow area in Fig. 9(b2) and use $Area_3$ and $Area_4$ to denote when $N = 3$ and 4, the horizontal line shadow region before $T_c$ in Fig. 8. We can establish the bounding relationship: $\frac{3}{4}Area_4 \leq Area_{3+} \leq \frac{4}{3}Area_3$. This can be generalized to $N$: $\frac{N}{N+1}Area_{N+1} \leq Area_{N+} \leq \frac{N+1}{N}Area_N$. The upper and lower bounds are proved. Please note that we purposely plot the fourth slot in Fig. 9(b2) slenderer than the first three as part of the production time in the fourth cycle is used to fill up the last production cycle ($T_c$) to make it the same size as in Fig. 9(b1) and 9(b3).

Next, we prove the monotonically decreasing property of the bounds. As we presented above, the lower bound of the grid line shadow in a general case is $\frac{N-1}{N}Area_N$ and the upper bound is $\frac{N+1}{N}Area_N$. As $N$ goes to infinity, both bounds reach $Area_N$. Without loss of generality, we use $Area_N$ in the following proof of the bounds property. In Fig. 8(b1), the horizontal line shadow area before $T_c$ is $Area_{N=1} = \frac{T_a^2}{2}$ and let $Area_{base} = \frac{T_a^2}{2}$; in Fig. 8(b2), the horizontal line shadow area is
Area_{N=2} = Area_{base} + \frac{T_a}{2} \frac{T_b}{2}; \text{ in Fig. 8 (b3), } Area_{N=3} = Area_{base} + \frac{T_a}{3} \frac{T_b}{3} + 2\frac{T_a}{3} \frac{T_b}{3}. \text{ To generalize the formula: } Area_N = Area_{base} + \sum_{i=1}^{N-1} \frac{T_a}{N} \frac{T_b}{N} = Area_{base} + \frac{T_a}{2} \frac{T_b}{N}, \text{ for all } N \geq 2. \text{ As } N \text{ increases, } Area_N \text{ increases, so the horizontal line shadow region size increases. The total age of the inventory is represented by the area of the tilted region, which is a fixed total region size minus the horizontal line shadow region; therefore it monotonically decreases as } N \text{ increases. As younger initial total age leads to larger revenue, the bound of the total revenue monotonically increases as } N \text{ increases.}

In Shen et al. (2011), they proved that the total cost at the boundary points (where } T_{inv} - T_c \text{ can be exactly divided into an integer number of EMQ cycles) is monotonically increasing beyond } \bar{N} \text{ where } \bar{N} \text{ is the smallest integer value satisfied by } Area_{cost}(N) - Area_{cost}(N + 1) \leq A, \text{ where } A \text{ is the fixed ordering cost and } Area_{cost}(N) \text{ is the total area for exactly } N \text{ triangles within } T_{triangle} = T_{inv} - T_c \text{ in the inventory plot Fig. 8(a1) - 8(a3). Total revenue increases at least } b\left[ \frac{T_a T_b}{2 N(N+1)} \right] \text{ as } N \text{ increases to } N + 1 \text{ and total cost increases } A - (Area_{cost}(N) - Area_{cost}(N + 1)) = A - \frac{D}{2} \left(1 - \frac{D}{P}\right) T_{triangle}^2 \frac{1}{N(N+1)} \text{ as } N \text{ increases to } N + 1, \text{ which implies the profit would change } \delta = \left[ b\left[ \frac{T_a T_b}{2 N(N+1)} \right] \right] \frac{1}{N(N+1)} - A. \text{ This proves that the total profit at the boundary points (where } T_{triangle} \text{ can be exactly divided into an integer number of EMQ cycles) is monotonically decreasing beyond } \bar{N}, \text{ where } \bar{N} \text{ is the smallest integer value satisfied by } \delta \leq 0. 

\section*{B Revenue Functions - Cases 2, 3, 4, and 5}

Let Revenue(Q) = x \cdot Q^2 + y \cdot Q + z \text{ for all cases. We continue using the notation in section 5 and below are notation we use in the appendix.}

\begin{itemize}
\item \(T_2:\) machine idle time in one regular production cycle
\item \(T_{p1}:\) time from the start of the additional production cycle to the end of the current underlying production cycle
\item \(T_{p3}:\) remainder of } T_{inv} \text{ divided by } T
\item \(I_{max}:\) maximum inventory level in a regular production cycle
\end{itemize}

\textbf{FIFO Policy}

\textbf{Case 2:}

\begin{align*}
x &= -\frac{bN^2(P-D)^2}{2PT_sD^2} - \left(1 - \frac{D}{P}\right) \frac{aN^2}{2T_sD} - \frac{N(1 - \frac{D}{P})a}{2T_sD} \\
y &= -\left[ \left( \frac{T_{inv} + \frac{T_{inv}(P-D)}{P}}{T_sD} \right) bN - \frac{N(P-D)bT_{inv}}{DPT_s} \right] (P-D) + \left(1 - \frac{D}{P}\right) aT_{inv}N - \frac{bN(P-D)}{D} - \frac{bT_{inv}N(P-D)^2}{PT_sD}
\end{align*}
\[ z = \left( \frac{T_{inv} + \frac{T_{inv}(P-D)}{p}}{T_s} \right) bT_{inv}(P - D) + \left( a - \frac{aT_{inv}}{T_s} \right) DT_{inv} - \frac{(1 - \frac{D}{p}) \alpha DT_{inv}^2}{2T_s} + \frac{bT_{inv}^2(P - D)^2}{2PT_s} + T_{inv}b(P - D) \]

Cases 3 and 4:

\[ x = -\frac{M(1 - \frac{D}{p})a}{2T_sD} + \frac{\alpha aM}{T_s} - \frac{(1 - \frac{D}{p})aM^2}{2T_sD} \]
\[ y = \left( \frac{1 - \frac{D}{p}}{T_s} \right) aT_{inv}M + M \left( a - \frac{T_{inv}a}{T_s} \right) - \left( a - \frac{(T_{inv} + \beta)a}{T_s} \right) M - \frac{\alpha aDT_{inv}}{T_s} \]
\[ z = \left( \frac{b(I_{min} - T_{inv}D)}{2PT_s} \right)^2 + \left( a - \frac{(T_{inv} + \beta)a}{T_s} \right) DT_{inv} - \frac{(1 - \frac{D}{p})\alpha DT_{inv}^2}{2T_s} + b(I_{min} - T_{inv}D) \]
\[ - \frac{\left( T_{inv} + \frac{I_{min} - T_{inv}D}{P}\right)b(I_{min} - T_{inv}D)}{T_s} \]

where \( \alpha = (M + 1) \left( \frac{1}{D} - \frac{1}{p} \right) \), and \( \beta = \frac{I_{min}}{P} - T_{inv} \).

Case 5:

\[ x = -\frac{(M - 1)(1 - \frac{D}{p})a}{2T_sD} - \left( \frac{-M^{-1} - \frac{1}{p} + \frac{M}{P}}{T_s} \right) bM - \frac{(1 - \frac{D}{p})a(M - 1)^2}{2T_sD} \]
\[ y = -\frac{(2T_{inv} - \frac{T_{inv}D}{P})bM}{T_s} + \left( \frac{-M^{-1} - \frac{1}{p} + \frac{M}{P}}{T_s} \right) bT_{inv}D + \frac{\left( T_{inv} + \frac{I_{min}}{P} \right)bM}{T_s} - \frac{bT_{inv}DM}{PT_s} + \frac{(1 - \frac{D}{p})aT_{inv}(M - 1)}{T_s} \]
\[ z = \frac{(2T_{inv} - \frac{T_{inv}D}{P})bT_{inv}D}{T_s} + \left( a - \frac{aT_{inv}}{T_s} \right) DT_{inv} - \frac{(1 - \frac{D}{p})\alpha DT_{inv}^2}{2T_s} + \frac{bT_{inv}^2D^2}{2PT_s} \]
\[ - \frac{\left( T_{inv} + \frac{I_{min}}{P} \right)bI_{min}}{T_s} - bT_{inv}D + \frac{bI_{min}^2}{2PT_s} + bI_{min} \]

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Case 2:

\[ x = -\frac{(1 + \frac{D}{p})aN^2P^2}{2T_sD^3} - \frac{a(N - 1)}{2DT_s} - \left( \frac{-N^2}{D^2} - \frac{N}{D} + \frac{3}{D} - \frac{1}{p} \right) aD \left( \frac{NP}{DS} - \frac{N-1}{D} \right) + \frac{(N - 1)aN}{T_sD} \]
\[ - \frac{(1 + \frac{D}{p})aD(\frac{NP}{DS} - \frac{N-1}{D})^2}{2T_s} + \frac{bN^2(P - D)^2}{2PT_sD^2} + \frac{(N - 1)aPN}{T_sD^2} + \frac{a(N - 1)}{2T_sP} - \frac{a(N - 1)^2}{DT_s} \]
\[
y = -\frac{(N-1)aT_{\text{inv}}}{T_s} - \frac{aNP}{D} + \frac{(1+\frac{D}{P}) aT_{\text{inv}}N P^2}{T_s D^2} + (N-1)a - \frac{(N-1)aPT_{\text{inv}}}{T_s D} - \frac{bN(P-D)}{D} \\
- \frac{(1+\frac{D}{P}) aD \left(T_{\text{inv}} - \frac{PT_{\text{inv}}}{D} \right) \left(\frac{NP}{D^2} - \frac{N-1}{D}\right)}{T_s} + \left[a - \left(\frac{PT_{\text{inv}}}{D} + T_{\text{inv}}\right) a\right] D \left(\frac{NP}{D^2} - \frac{N-1}{D}\right) \\
- \frac{(-NP - \frac{N}{D} + \frac{3}{P} - \frac{1}{P}) aD \left(T_{\text{inv}} - \frac{PT_{\text{inv}}}{D} \right) + 2T_{\text{inv}} bN(P-D) + \frac{bT_{\text{inv}}N(P-D)^2}{PT_s D}}{T_s} \\
+ \left[a - \left(\frac{PT_{\text{inv}}}{D} + T_{\text{inv}}\right) a\right] D \left(\frac{NP}{D^2} - \frac{N-1}{D}\right) - \frac{(1+\frac{D}{P}) aD \left(T_{\text{inv}} - \frac{PT_{\text{inv}}}{D} \right)^2}{2T_s}
\]

Cases 3 and 4:

\[
x = -\frac{(1+\frac{D}{P}) aD \left[\frac{-(N-M)P}{D^2} - \frac{L}{D}\right]^2}{2T_s} - \frac{(1+\frac{D}{P}) a (N-M) P^2}{2T_s D^3} + \frac{LaM}{T_s D} - \frac{b}{2PT_s} \\
- \frac{\left[\frac{(N-M)P}{D^2} + \frac{3L}{D} - \frac{M-1}{D} - \frac{1}{P}\right] aD \left[\frac{-(N-M)P}{D^2} - \frac{L}{D}\right]}{2T_s D^2} - \frac{aL}{2DT_s} + \frac{aL}{2T_s P} - \frac{LaPN}{T_s D^2} \\
+ \frac{LaPM}{T_s D^2} + \frac{b \left[1 - \left[\frac{-(N-M)P}{D^2} - \frac{L}{D}\right] D\right]^2}{2PT_s} + \frac{b \left[1 - \left[\frac{-(N-M)P}{D^2} - \frac{L}{D}\right] D\right]}{DT_s} + \frac{b \left[1 - \left[\frac{-(N-M)P}{D^2} - \frac{L}{D}\right] D\right]}{DT_s} \\
+ \frac{b (K+1)}{2PT_s} - \frac{aL^2}{DT_s} - \frac{3b (K+1)}{2DT_s} + \frac{b (K+1)^2}{2DT_s}
\]

\[
y = b (K+1) + \left(a - \left[\frac{T_{\text{inv}} + \left(\frac{PT_{\text{inv}} + T_{p_1}}{D}\right)}{T_s}\right] a\right) D \left[\frac{-(N-M)P}{D^2} - \frac{L}{D}\right] - \frac{2b (K+1) T_{\text{inv}}}{T_s}
\]

\[
- \frac{\left[\frac{(N-M)P}{D^2} + \frac{3L}{D} - \frac{M-1}{D} - \frac{1}{P}\right] aD \left[T_{\text{inv}} - \frac{(T_{p_1} + T_{p_3})P}{D}\right]}{T_s} + b \left[1 - \left[\frac{-(N-M)P}{D^2} - \frac{L}{D}\right] D\right] \\
- \frac{(1+\frac{D}{P}) a \left(T_{p_1} + T_{p_3}\right) (N-M) P^2}{T_s D^2} - \left[\frac{T_{\text{inv}} - \frac{(T_{p_1} + T_{p_3})P}{D}}{T_s}\right] D \left[\frac{-(N-M)P}{D^2} - \frac{L}{D}\right] D \\
- \frac{2T_{\text{inv}} b \left[1 - \left[\frac{-(N-M)P}{D^2} - \frac{L}{D}\right] D\right]}{T_s} - \frac{Kb \left[T_{\text{inv}} - \frac{(T_{p_1} + T_{p_3})P}{D}\right]}{T_s} - \frac{LaPT_{p_1}}{T_s D} - \frac{LaT_{\text{inv}}}{T_s} + \frac{La}{T_s D} \\
- \frac{a \left[T_{\text{inv}} - \frac{(T_{p_1} + T_{p_3})P}{D}\right]}{T_s} D + \frac{2T_{\text{inv}} b \left[T_{\text{inv}} - \frac{(T_{p_1} + T_{p_3})P}{D}\right]}{T_s} D
\]

\[
z = -b \left[T_{\text{inv}} - \frac{(T_{p_1} + T_{p_3})P}{D}\right] D + \frac{2T_{\text{inv}} b \left[T_{\text{inv}} - \frac{(T_{p_1} + T_{p_3})P}{D}\right]}{T_s} D \\
+ \left(a - \left[\frac{T_{\text{inv}} + \left(\frac{PT_{\text{inv}} + T_{p_1}}{D}\right)}{T_s}\right] a\right) D \left[T_{\text{inv}} - \frac{(T_{p_1} + T_{p_3})P}{D}\right] + \frac{b \left[T_{\text{inv}} - \frac{(T_{p_1} + T_{p_3})P}{D}\right]^2}{2PT_s D^2}
\]

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\[
\frac{(1 + \frac{p}{P}) a D \left[ T_{\text{inv}} - \frac{(T_{p1} + T_{p3})P}{D} \right]^2}{2T_s} - \frac{(1 + \frac{p}{P}) a (T_{p1} + T_{p3})^2 P^2}{2T_s D} + a (T_{p1} + T_{p3}) P
\]

where \( \alpha = (M + 1) \left( \frac{1}{T} - \frac{1}{P} \right) \), \( \beta = \frac{L_{\text{inv}}}{P} - T_{\text{inv}} \), \( L = \left[ \frac{T_{\text{inv}} - T_f}{T} \right] \), \( K = \left[ \frac{L_{\text{inv}} - D T_{\text{inv}}}{Q} \right] \), and \( T_f = (T_{p1} + (N - M)T + T_{p3}) \frac{P}{D} \).

Case 5:

\[
x = -\frac{a L^2}{DT_s} + \frac{b}{DT_s} \frac{a L P \alpha}{T_s D} - \frac{b}{DT_s} - \frac{2 b T_s}{2PT_s} + \frac{b}{2PT_s} + \frac{La M + La P M}{T_s D} + \frac{a L}{2T_s P} + \frac{b(K + 1)}{2PT_s} \left[ (\alpha + \frac{N - M}{D}) - \frac{N}{D} \right] P \frac{a D}{T_s} \left[ \frac{(\alpha + \frac{N - M}{D}) - \frac{N}{D}}{T} - \frac{L}{D} \right] - 3b(K + 1)
\]

\[
y = -\frac{b \left[ T_{\text{inv}} - \frac{(T_{\text{inv}} + \beta)P}{D} \right] D \left( 1 - \left[ (\alpha + \frac{N - M}{D}) - \frac{N}{D} \right] P \frac{a D}{T_s} \left[ \frac{(\alpha + \frac{N - M}{D}) - \frac{N}{D}}{T} - \frac{L}{D} \right] \right) - \frac{La P T_{\text{inv}}}{T_s D} + La + b(K + 1)
\]

\[
y + b \left( 1 - \left[ (\alpha + \frac{N - M}{D}) - \frac{N}{D} \right] P \frac{a D}{T_s} \left[ \frac{(\alpha + \frac{N - M}{D}) - \frac{N}{D}}{T} - \frac{L}{D} \right] \right) - \frac{(1 + \frac{P}{P}) a (T_{\text{inv}} + \beta) \left[ (\alpha + \frac{N - M}{D}) - \frac{N}{D} \right] P^2}{T_s D}
\]

\[
y + a \left[ -\left( \frac{T_{\text{inv}} + (T_{\text{inv}} + \beta)P}{T_s} \right) a D \left[ \frac{(\alpha + \frac{N - M}{D}) - \frac{N}{D}}{T} - \frac{L}{D} \right] \right] - \frac{La P \beta}{T_s D} \left[ \frac{(\alpha + \frac{N - M}{D}) - \frac{N}{D}}{T} - \frac{L}{D} \right]
\]

\[
y - \frac{\left( \frac{(\alpha + \frac{N - M}{D}) - \frac{N}{D}}{T} + \frac{2 L}{D} - \frac{M - 1}{P} \right) a D \left[ T_{\text{inv}} - \frac{(T_{\text{inv}} + \beta)P}{D} \right]}{T_s} - \frac{La T_{\text{inv}}}{T_s} + \frac{2 T_{\text{inv}} b}{T_s}
\]

\[
y + \frac{(1 + \frac{P}{P}) a D \left[ T_{\text{inv}} - \frac{(T_{\text{inv}} + \beta)P}{D} \right]}{T_s} \left[ \frac{(\alpha + \frac{N - M}{D}) - \frac{N}{D}}{T} - \frac{L}{D} \right] + \frac{(1 + \frac{P}{P}) a D \left[ T_{\text{inv}} - \frac{(T_{\text{inv}} + \beta)P}{D} \right]}{2T_s}
\]

\[
y + \frac{(1 + \frac{P}{P}) a D \left[ T_{\text{inv}} - \frac{(T_{\text{inv}} + \beta)P}{D} \right]^2}{2T_s} - b \left[ T_{\text{inv}} - \frac{(T_{\text{inv}} + \beta)P}{D} \right] D
\]

\[\text{27}\]
\[-\frac{(1 + \frac{D}{P}) a (T_{inv} + \beta)^2 P^2}{2T_s D} + a(T_{inv} + \beta) P + \frac{b \left[T_{inv} - \frac{(T_{inv} + \beta) P}{D}\right]^2 D^2}{2PT_s}\]

where \( T_f = \frac{-I_{max} + (T_{inv} - NT - T_1)D + (I_{max} - DT_{inv})}{D} \), \( L = \left\lfloor \frac{T_{inv} - T_f}{T} \right\rfloor \), and \( K = \left\lfloor \frac{I_{min} - DT_{inv}}{Q} \right\rfloor \).