

# LOANS FOR WHICH THE ACQUIRED ASSET SERVES AS THE COLLATERAL

GERALD A. FLEISCHER AND MAGED M. DESSOUKY  
University of Southern California

SHU MING NG  
Hong Kong University of Science and Technology

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## ABSTRACT

This paper addresses the loan environment in which (a) the asset funded by the loan serves as collateral for the loan, (b) the funded asset decreases in value over the period of the loan at a known constant rate, and (c) the loan is repaid in a finite number of equal end-of-period payments. Of particular concern are the conditions under which the value of the collateral may be less than the unpaid principal during the loan period, thereby creating an unwelcome situation for the lender. The loan interest rate is determined such that the unpaid loan principal is at all times equal to or smaller than the current market value of the collateral.

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## INTRODUCTION

Investors frequently fund the acquisition of certain assets by borrowing a portion of the initial cost with repayment over a fixed number of periods. Usually, the acquired asset serves as the collateral for the loan. Examples are consumer loans for the purchase of automobiles and industrial loans for the purchase of new equipment. From the lender's point of view, it is important that the (market) value of the collateral throughout the period of the loan never be less than the remaining principal owed on the loan.

The unpaid principal at any point in time depends upon the amount of the original loan, the loan repayment plan, the length of the loan, and the interest rate on the loan. There are certain combinations of these factors where the unrecovered principal is at least as large as the collateral value. When this situation occurs, an opportunity exists for the borrower to renege on the loan, thereby lowering the real return to the lender. This situation is illustrated in Figure 1.

Assuming that the collateral value is a monotone decreasing function over time, a typical collateral value function is represented by curve A-A. The three B-B curves are sample plots of the remaining principal on the loan assuming equal periodic payments. The *lower* curve reflects a loan repayment plan when the interest rate is zero. Thus, each equal loan payment is repayment of

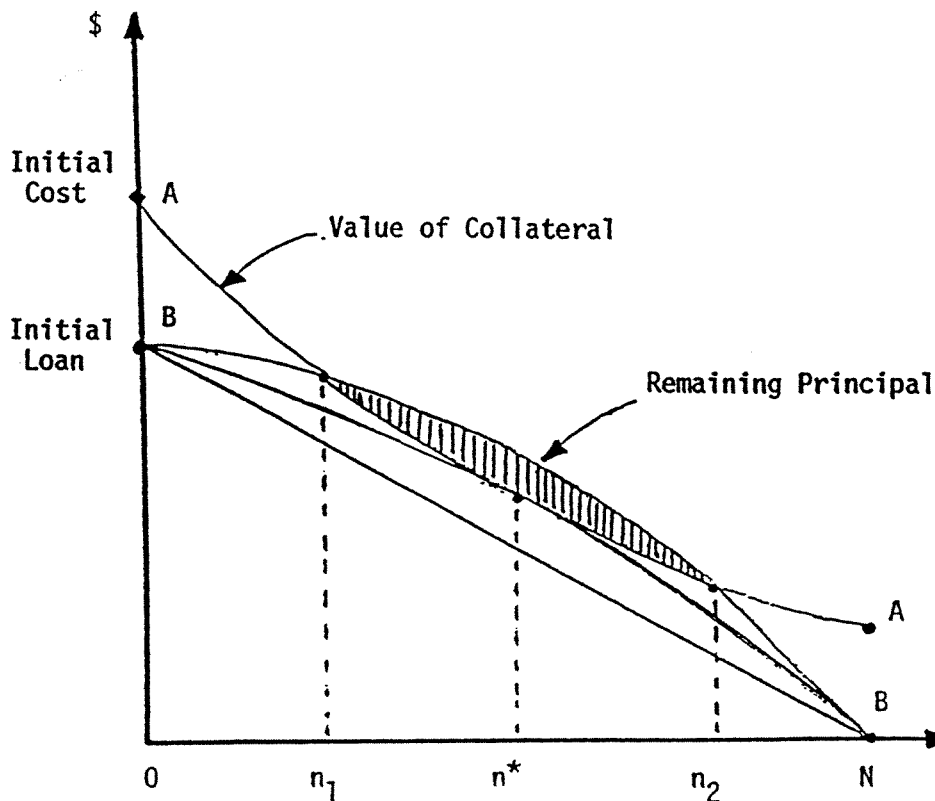


FIGURE 1. Declining value of collateral and reduced outstanding loan principal.

principal only. Both the *middle* and *upper* B-B curves represent a loan repayment plan when the interest rate is greater than zero. The interest rate resulting in the upper curve is higher than that for the middle curve. The loan payment plan in both these cases consists of both a repayment of principal and interest. With each successive payment, the principal portion increases while the interest portion decreases. Hence, the remaining principal is a concave function assuming a nonnegative interest rate.

Note that in the *upper* B-B curve there is a time interval ( $n_1, n_2$ ) during which the remaining principal on the loan is greater than the collateral value. This situation from the lender's viewpoint is undesirable because the borrower is

more inclined to default on the loan. As Waller [12] points out, the most important factor of default is the equity position of the borrower.

The *middle* B-B curve is tangent to the A-A curve at  $n^*$ . The point of tangency defines the maximum possible loan interest rate such that the unrecovered principal is never greater than the collateral value throughout the period of the loan. Since the loan repayment plan depends on the interest rate, the problem for the lender is to determine the maximum value of this interest rate such that the time interval  $(n_1, n_2)$  has length zero ( $n_2 - n_1 = 0$ ).

There is extensive research in the determination of the loan value especially mortgages when loan prepayment is allowed (Brennan and Schwartz [3], Dunn and McConnell [4], Green and Shoven [6], Quigley [9], Schwartz and Torous [10]). This earlier work stems from the notion that borrowers will reassess their loans in view of declining interest rates, and thus there exists a probability of prepayment at the end of each payment period. Schwartz and Torous [11] extend the work by including the possibility of default in determining the value of the loan. Another performance measure of a mortgage loan is the duration which measures the ratio of the change in loan value to the change in interest rate. Haensly, Springer, and Waller [7] determine the duration for a fixed-rate mortgage assuming a continuous payment plan. The above models do not determine the largest *interest rate* that can be charged that insures that the value of the asset will be greater than the remaining unpaid balance during the life of the loan. The models assume that the interest rate is given. Benston, Horsky, and Weingartner [2] determine, for a given interest rate, the maximum *loan-to-value ratio* in which the collateral value is not less than the remaining principal by solving a linear equation relating the collateral value to the remaining principal. In this paper, we determine the largest interest rate such that the unpaid loan principal is at all times equal to or smaller than the value of the collateral given a loan-to-value ratio and rate of decline of collateral value.

It should be noted that, as a practical matter, the lender does not set the interest rate in a vacuum. Interest rates are influenced by a host of factors, not the least of which is aggressive competition. However, by calculating the maximum interest rate that defines the point of tangency between the remaining principal curve and the collateral value curve, the lender can determine if undesirable conditions are possible under current or expected future loan interest rates.

#### MATHEMATICAL MODEL

The notation for the parameters and computed variables is as follows:

##### *Parameters*

- $C$  initial cost (value) of the asset.
- $d$  rate of decline in asset value per period,  $0 \leq d \leq 1.0$ , assumed constant over the life of the loan.

- $P$  amount of the initial loan,  $P \leq C$ .  
 $N$  number of end-of-period payments.  
 $\rho$  ratio of loan amount to initial cost,  $\rho = P/C$ . This is known as the "loan-to-value ratio".  
 $n$  discrete time period,  $n = 1, 2, \dots, N$ .  
 $i$  loan interest rate per period,  $i \geq 0$ , assumed constant over the life of loan.

#### Computed Variables

- $A$  amount of equal end-of-period loan payment.  
 $F_1(n)$  net market value of the asset at the end period  $n$ .  
 $F_2(n, i)$  remaining principal after  $n$  periods given interest rate  $i$ .  
 $i^*$  maximum interest rate such that  $F_1(n) \geq F_2(n, i)$  for all  $n \in (0, N]$ .

There are a wide variety of functional forms for  $F_1(n)$  but for the purpose of this discussion the following form is assumed:

$$F_1(n) = C(1 - d)^n \quad (1)$$

The function  $F_1(n)$  is represented by curve A-A in Figure 1. Assuming that the initial loan amount ( $P$ ) is repaid by  $N$  equal end-of-period payments of amount  $A$ , the relationship between  $A$  and  $P$  is:

$$A = P \left[ \frac{i(1+i)^N}{(1+i)^N - 1} \right] \quad (2)$$

The remaining principal after  $n$  such payments is:

$$F_2(n, i) = A \left[ \frac{i(1+i)^{N-n} - 1}{i(1+i)^{N-n}} \right] = P \left[ \frac{(1+i)^N - (1+i)^n}{(1+i)^N - 1} \right] \quad (3)$$

The function  $F_2(n, i)$  is represented by curve(s) B-B in Figure 1.

The duration of the "time period" in the above relationships is usually specified in the terms of the loan. Loan rates are typically specified per annum and loans are frequently paid in monthly payments. The rate of decline of collateral value,  $d$ , can be estimated either monthly or annually. Thus, it is useful to be able to convert rates from per annum to per month, or vice versa. For example:

$$i_a = (1 + i_m)^{12} - 1 \quad (4)$$

$$d_a = 1 - (1 - d_m)^{12} \quad (5)$$

The subscripts  $a$  and  $m$  indicate per annum and per month, respectively.

As an example, consider the acquisition of an asset at an initial cost ( $C$ ) of \$25,000 with rate of depreciation ( $d$ ) estimated to be 2% per month. (Here, "depreciation" refers to loss in market value rather than decline in book value.) The lender loans 80% of the initial cost ( $P = \$20,000$ ) at an effective interest rate of 1.0% per month over 84 months. Figure 2 plots the resulting  $F_1(n)$  and  $F_2(n, i)$  functions. As shown in Figure 2,  $F_1(n) = F_2(n, i)$  at  $n_1 \approx 21$  and  $n_2 \approx 60$  months. That is, the unpaid remaining principal exceeds the collateral value from about the 21st to the 60th month. From the lender's point of view, this undesirable condition would not have arisen if the loan interest rate had been suitably less than 1.0% per month.

As previously noted, the problem is to find the maximum value of  $i$ , say  $i^*$ , such that the market value of the collateral is at least as large as the unpaid principal at any time during the loan repayment period. That is, given values for the parameters  $\rho = P/C$ ,  $d$ , and  $N$ , maximize  $i$  subject to:

$$\text{Constraint 1: } F_1(n) - F_2(n, i) \geq 0, \text{ for all } n \in (0, N]$$

$$\text{Constraint 2: } i \geq 0$$

Constraint 1 ensures that the value of the collateral is greater than the remaining principal at all values of  $n$  during the length of the loan  $N$ . The problem is formulated with  $n$  as a continuous variable because of computational advantages that will be explained below. At the maximum interest rate  $i^*$ , if constraint 1 is feasible for all  $n \in (0, N]$ , then constraint 1 will be feasible for any  $n = 1, 2, \dots, N$ . Constraint 2 ensures that the maximum interest rate is non-negative.

The above formulation is a constrained nonlinear optimization problem. These problems are typically very difficult to solve (Luenberger [8]). However, we will make use of special properties of the functions  $F_1(n)$  and  $F_2(n, i)$  to solve the problem using unconstrained nonlinear optimization solution techniques. An equivalent formulation is to find the tangency point  $(n^t, i^t)$  of the functions  $F_1(n)$  and  $F_2(n, i)$ . The tangency point  $i^t$  is equal to the maximum interest rate  $i^*$  (critical interest rate) of the nonlinear constrained optimization problem because constraint 1 will be violated at higher interest rate values, as illustrated in Figure 1. Let the function  $G(n, i)$  be defined as follows:

$$G(n, i) = \frac{(F_1(n) - F_2(n, i))}{C} \quad (6)$$

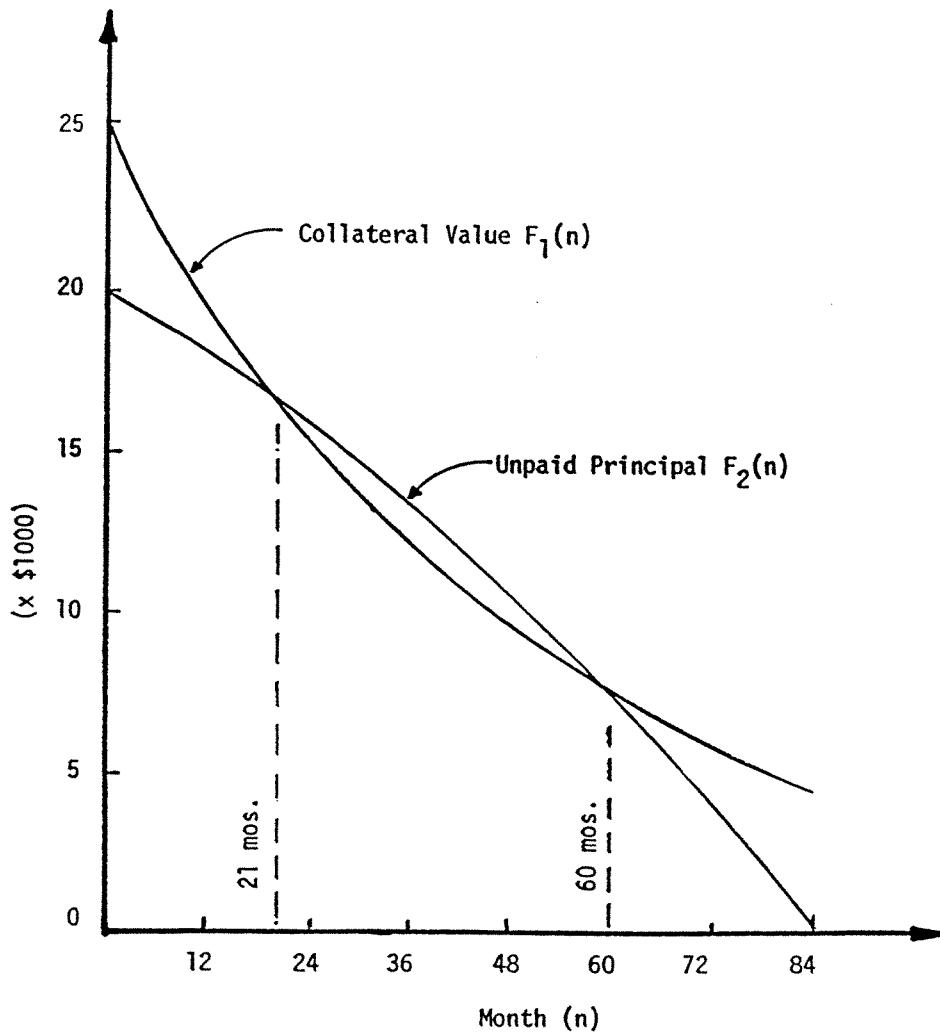


FIGURE 2. Collateral value and outstanding debt (unpaid principal).

By fixing  $n$  and increasing  $i$ , the function  $G(n, i)$  strictly decreases. Hence, multi-dimensional linear search rules may be used to find  $i^t$  where  $G(n, i^t) = 0$  at only  $n = n^t$ . The value for  $i^t$  is searched for in the range  $[0.0, 1.0]$  and  $n^t$  in the range  $(0, N]$ . The above formulation has  $n$  as a continuous variable instead of a discrete variable because it is easier to perform a continuous search for  $n^t$ . This search makes use of unconstrained optimization solution techniques and is described in the Appendix.

## SAMPLE PROBLEMS AND DISCUSSION

To gain insight into the problem, the critical interest rate  $i^*$  is calculated such that the value of the collateral is not less than the remaining principal during the length of the loan for selected values of  $N$ ,  $\rho$ , and  $d$ . The tested input values are:

- $N = 12, 24, 36, 48, 60, 72, 84, 96$  months
- $d_m = 0.010, 0.015, 0.020, 0.025, 0.030$
- $\rho = 0.50, 0.55, 0.60, \dots, 0.90, 0.95$

The solution set for the critical interest rate  $i^*$  per year is summarized in Table 1. The term "all" appears in the body of the table to indicate that the critical interest rate  $i^*$  is at least 100%. That is, the collateral value will exceed the outstanding principal for all practical loan rates. The term "none" indicates that there is no non-negative loan interest rate in which the collateral value will exceed the outstanding principal for the entire duration of the loan. Clearly for this case it is not advantageous for the lender to loan the money.

Table 1 provides guidance for lenders by indicating the circumstances under which the condition  $F_1(n) < F_2(n, i)$  occurs. For example, suppose that the lender is considering a 60-month loan on a certain asset for which it is believed that the collateral value will decrease by 2% per month. The borrower wishes to borrow 85% of the initial cost of the asset. If the effective loan interest rate is 15% per annum, will there be any time during the 60-month period at which the collateral value will be less than the unpaid principal? Entering Table 1 for  $N = 60$ ,  $d_m = 0.02$ , and  $\rho = 0.85$ , we find that  $i^* = 24.6\%$  which is well above the effective interest rate of 15%. Thus, the collateral value is expected to be above the remaining principal throughout the length of the loan. Note that an increase of  $\rho$  to 0.90 would result in a critical value ( $i^* = 15.2\%$ ) very close to the effective loan interest rate.

To provide further insight into the parameter relationships, selected  $i^*$  values are illustrated in Figure 3 for  $N = 36, 72$ ;  $\rho = 0.75, \dots, 0.95$ ; and  $0 \leq d \leq 0.50$ . From Table 1 and Figure 3, a number of conclusions are apparent:

- 1) For a specified loan payment period ( $N$ ) and loan percentage ( $\rho$ ), the critical value of the loan rate ( $i^*$ ) decreases as the collateral depreciation rate ( $d$ ) increases. Larger values of  $d$  represent a more rapid decline in collateral value over time, i.e., a smaller market value. For this reason, the critical value of the loan interest rate must be smaller in order to assure that the collateral value is not exceeded by the remaining principal.

TABLE 1. Maximum annual interest rate(%),  $i^*$ ,  
for selected values of  $N$ ,  $P/C$ , and  $d$ .

loan duration months ( $N$ )	loan/asset ratio ( $\rho$ )	monthly depreciation ( $d$ %)				
		1.0	1.5	2.0	2.5	3.0
12	0.50 - 0.95	all	all	all	all	all
24	0.50 - 0.95	all	all	all	all	all
36	0.95	all	all	71.85	41.57	20.66
	0.90	all	all	all	68.70	39.97
	0.85	all	all	all	all	61.75
	0.80	all	all	all	all	88.57
	0.50 - 0.75	all	all	all	all	all
48	0.95	97.13	50.34	24.70	7.59	none
	0.90	all	77.10	41.19	19.13	3.62
	0.85	all	all	59.72	31.39	12.52
	0.80	all	all	92.61	45.54	22.39
	0.75	all	all	all	62.81	33.88
	0.70	all	all	all	85.07	47.80
	0.65	all	all	all	all	65.45
	0.60	all	all	all	all	89.00
	0.50 - 0.55	all	all	all	all	all
60	0.95	52.59	23.52	6.17	none	none
	0.90	78.87	37.29	15.20	0.51	none
	0.85	all	52.66	24.62	7.12	none
	0.80	all	71.61	35.28	14.29	none
	0.75	all	96.89	48.02	22.42	5.70
	0.70	all	all	64.06	32.01	12.32
	0.65	all	all	85.49	43.77	20.08
	0.60	all	all	all	58.84	29.48
	0.55	all	all	all	79.33	41.30
	0.50	all	all	all	all	56.95
72	0.95	31.38	10.21	none	none	none
	0.90	46.58	18.76	2.67	none	none
	0.85	64.55	27.74	8.49	none	none
	0.80	88.49	38.02	14.78	0.05	none
	0.75	all	50.33	21.90	4.90	none
	0.70	all	66.69	30.26	10.37	none
	0.65	all	89.09	40.51	16.71	1.42
	0.60	all	all	53.68	24.30	6.61
	0.55	all	all	71.64	33.73	12.77
	0.50	all	all	98.33	46.01	20.32
84	0.95	19.38	2.55	none	none	none
	0.90	29.42	8.47	none	none	none
	0.85	40.52	14.45	none	none	none
	0.80	54.09	21.00	3.84	none	none
	0.75	72.17	28.56	8.47	none	none
	0.70	98.69	37.70	13.71	none	none
	0.65	all	49.28	19.82	3.23	none
	0.60	all	64.92	27.21	7.91	none
	0.55	all	87.88	36.51	13.46	none
	0.50	all	all	48.87	20.25	3.58
96	0.95	11.85	none	none	none	none
	0.90	19.05	2.09	none	none	none
	0.85	26.67	6.42	none	none	none
	0.80	35.50	11.02	none	none	none
	0.75	46.45	16.16	0.55	none	none
	0.70	61.04	22.11	4.20	none	none
	0.65	82.27	29.28	8.33	none	none
	0.60	all	39.30	13.15	none	none
	0.55	all	50.32	18.94	2.41	none
	0.50	all	67.65	26.17	6.82	none

- 2) For a specified loan payment period ( $N$ ) and collateral depreciation rate ( $d$ ), the critical value ( $i^*$ ) increases as the loan-to-asset-value ratio ( $\rho$ ) decreases. If the original loan represents a small percentage of the acquisition cost of the asset, the lender can charge a higher loan rate while ensuring that the remaining principal is below the collateral value.
- 3) The critical value of the loan rate ( $i^*$ ) is also inversely proportional to the loan repayment period ( $N$ ). For given values of  $\rho$  and  $d$ ,  $i^*$  decreases as  $N$  increases.
- 4) Review of Table 1 indicates that there are relatively limited number of combinations of  $N$ ,  $d$  and  $\rho$  for which there exists a nonnegative value of  $i^*$ . The "none" entries in Table 1 indicate that, for the specified combination of parameter values, the collateral value will be lower than the remaining principal at some time during the loan repayment period. This condition typically occurs when:
  - a) Given  $N$  and  $\rho$ ,  $d$  is relatively large,
  - b) Given  $N$  and  $d$ ,  $\rho$  is relatively large, and
  - c) Given  $d$  and  $\rho$ ,  $N$  is relatively large.
- 5) The desirable effect,  $F_1(n) \geq F_2(n, i)$  for values of  $n$  in the range  $(0, N]$ , can be obtained under certain limited conditions. If there is no reduction in collateral value ( $d \leq 0$ ), the remaining principal obviously must be less than the collateral value for all  $i \geq 0$  and all  $1 \leq n \leq N$ . In general,  $F_1(n) \geq F_2(n, i)$  for values of  $n$  in the range  $(0, N]$  when:
  - a) Given  $N$  and  $\rho$ ,  $d$  is relatively small,
  - b) Given  $N$  and  $d$ ,  $\rho$  is relatively small, and
  - c) Given  $d$  and  $\rho$ ,  $N$  is relatively small.
- 6) Both Table 1 and Figure 3 indicate the extent to which the critical value  $i^*$  is remarkably sensitive to the relevant values of the parameters  $\rho$ ,  $d$ , and  $N$ . Relatively small changes in any one of these three parameters can affect  $i^*$  significantly. For example, for  $\rho = 0.95$  and  $N = 48$ , desirable conditions from the lender's viewpoint appear reasonably assured when  $d_m = 0.02$  since the resulting  $i^*$  is equal to 24.7%. If the monthly depreciation rate is slightly higher, say  $d_m = 0.025$ , the critical value falls to 7.5%, a value which is likely to be less than the loan interest rate. In those instances where there is a high degree of uncertainty concerning the depreciation rate, the lender might be better advised to shorten the loan repayment period or to decrease the loan percentage,  $\rho$ .

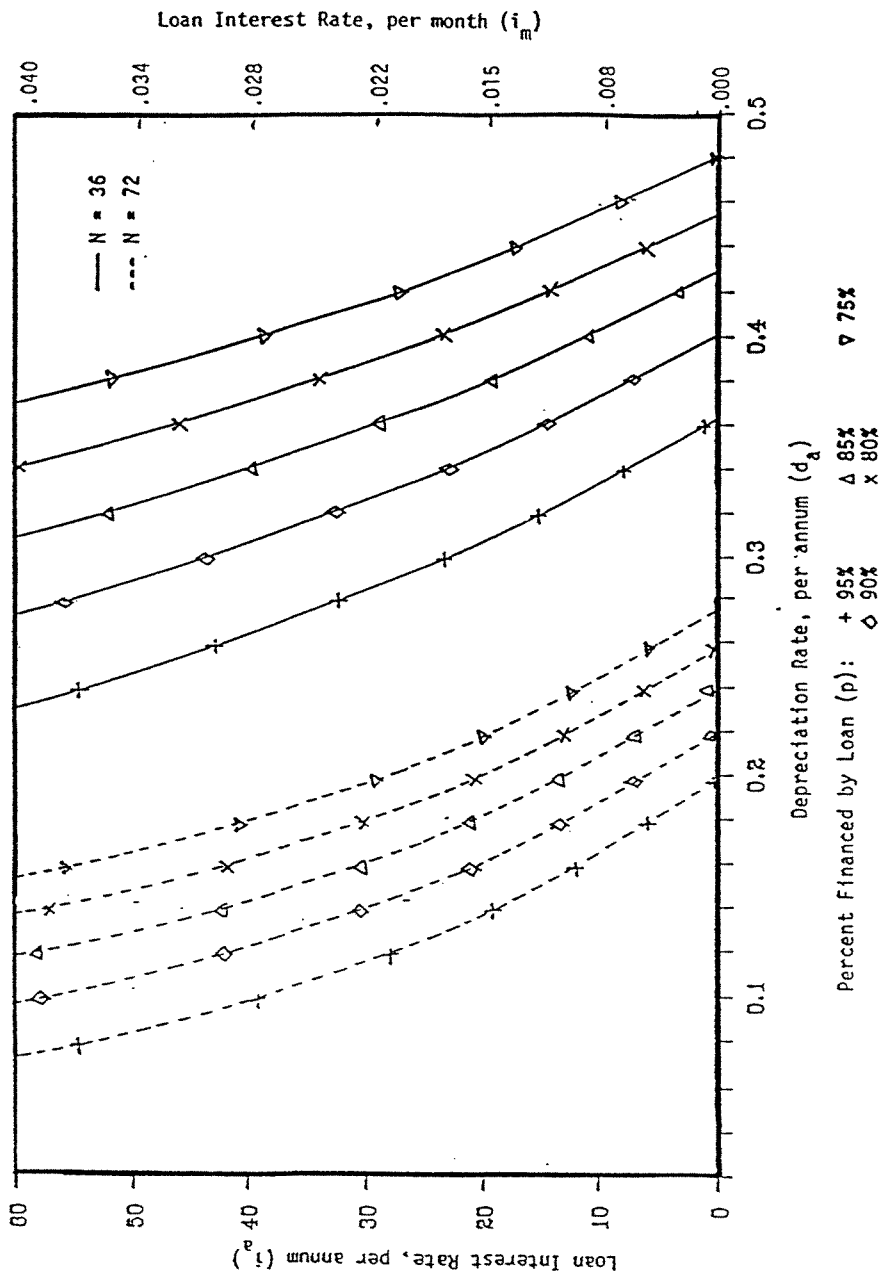


FIGURE 3. Maximum loan interest rate to insure that collateral value will exceed outstanding principal over the life of the loan.

## CONCLUSIONS

A nonlinear programming problem is formulated to determine the maximum interest rate that ensures the value of the collateral to not be less than the remaining principal on the loan during the length of the loan. Some general conditions that usually ensure that the value of the collateral is at no time less than the remaining principal are: (a) the asset has a small depreciation rate; (b) the amount of the loan is a small fraction of the original value of the asset; and (c) the loan repayment period is relatively short. Although these three factors may be understood intuitively, their interrelationships, and their relation to the critical loan interest rate can only be evaluated fully through analysis of the type described in this paper.

It should be noted that it might be possible that the lender can get a higher return by charging a higher interest rate than the critical interest rate and taking the risk of possible default, especially if the default occurs in later periods. Future research can study the relationship between the default risk and the rate of return to the lender. Other areas of future research include studying the effects of uncertainty in future interest rates and depreciation on the critical interest rate.

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#### APPENDIX

This appendix describes an unconstrained optimization solution procedure to calculate the tangency point  $(n^t, i^t)$  of functions  $F_1(n)$  and  $F_2(n, i)$ . As previously discussed,  $i^t$  is equal to the critical interest rate value  $i^*$ . The standard linear search problem is to minimize some function, say  $H(x)$ , subject to  $a \leq x \leq b$ . The interval  $[a, b]$  is referred to as the *interval of uncertainty* since the exact location of the minimum point is not known. The main idea of the search procedure is to reduce the interval of uncertainty by excluding portions of the interval that do not contain the minimum. The procedure terminates when the interval of uncertainty is sufficiently small; the minimum of  $H(x)$  is usually identified to be in the middle of the interval. This procedure works only if  $H(x)$  is a convex function (Bazaraa and Shetty [1] and Luenberger [8]).

Let the function  $G(n, i)$  be defined as follows:

$$G(n, i) = \frac{(F_1(n) - F_2(n, i))}{C} \quad (A1)$$

Since the function  $G(n, i)$  is a function of two variables,  $n$  and  $i$ , multi-dimensional search rules based on the "cyclic coordinate method" are used to find  $i^t$  where  $G(n, i^t) = 0$  at only  $n = n^t$ . The value for  $i^t$  is searched in the range  $[0.0, 1.0]$  and  $n^t$  in the range  $(0, N]$ .

The cyclic coordinate method changes one variable in the search while keeping all other variables fixed. Thus, we alternate between searching in the direction of  $n$  given a fixed interest rate, say  $\bar{i}$ , for the minimum of the function  $G(n, \bar{i})$  and searching in the direction  $i$  given a fixed period, say  $\bar{n}$ , for the minimum of the function  $G(\bar{n}, i)$ .

The "golden section method" is used to perform a one-dimensional linear search to find the point  $(n', \bar{i})$  in which the function  $G(n, \bar{i})$  is a minimum over a closed bounded interval  $[0, N]$ . The golden section method does not use derivatives for minimizing the function  $G(n, \bar{i})$ . It is a sequential search procedure that uses the values of the previous iterations to determine the succeeding readings. The golden search method is one of the most efficient non-derivative line search techniques.

The one-dimensional search for the interest rate  $i'$  in which the function  $G(\bar{n}, i) = 0$  is based on the "bisection search method". This method is a derivative-based search method. The bisection search method is selected because the determination of  $G'(\bar{n}, i)$  is relatively straight forward. The reason that we search for the minimum of function  $G(n, \bar{i})$  and the point where  $G'(\bar{n}, i) = 0$  will be explained later. First, the functions  $G(n, \bar{i})$  and  $G(\bar{n}, i)$  are shown to be convex since both the golden search and bisection search methods require that the function be convex.

*Proposition 1:* For a fixed  $\bar{i}$ , the function  $G(n, \bar{i})$  is convex.

*Proof:* A formal proof is omitted; however, an outline of the proof is provided. The function  $G(\bar{n}, i)$  can be written as follows:

$$G(n, \bar{i}) = D_1^n + \frac{(1+i)^n}{D_2} - D_3 \quad (\text{A2})$$

where  $D_1, D_2$ , and  $D_3$  are positive constants. Note that  $G(n, \bar{i})$  is the sum of two convex functions. Hence,  $G(n, \bar{i})$  is also convex.

*Proposition 2:* For a fixed  $\bar{n}$ , the function  $G(\bar{n}, i)$  is convex.

*Proof:* A formal proof is omitted; however, an outline of the proof is provided. The function  $G(\bar{n}, i)$  can be written as follows:

$$G(\bar{n}, i) = D_1 - \rho \left[ \frac{(1+i)^N - (1+i)^n}{(1+i)^N - 1} \right] \quad (\text{A3})$$

where  $D_1$  is a positive constant. Note that  $G(\bar{n}, i)$  is directly related to the negative of a concave function. Hence,  $G(\bar{n}, i)$  is also convex.

An outline of the solution procedure is shown in Figure A1 and the solution procedure can be summarized as follows:

#### STEP 0

- 0.1 Select  $\epsilon$ ,  $\delta$ , and  $I$ .
- 0.2 Find the minimum of  $G(n, 0)$  using the golden section method. Let the minimum solution be  $(n', 0)$ .
- 0.3 If  $G(n', 0) \leq 0 \Rightarrow$  stop and conclude no positive interest rate is chargeable.  
If  $G(n', 0) > 0 \Rightarrow$  set  $i^u = 0$  and go to step 1.

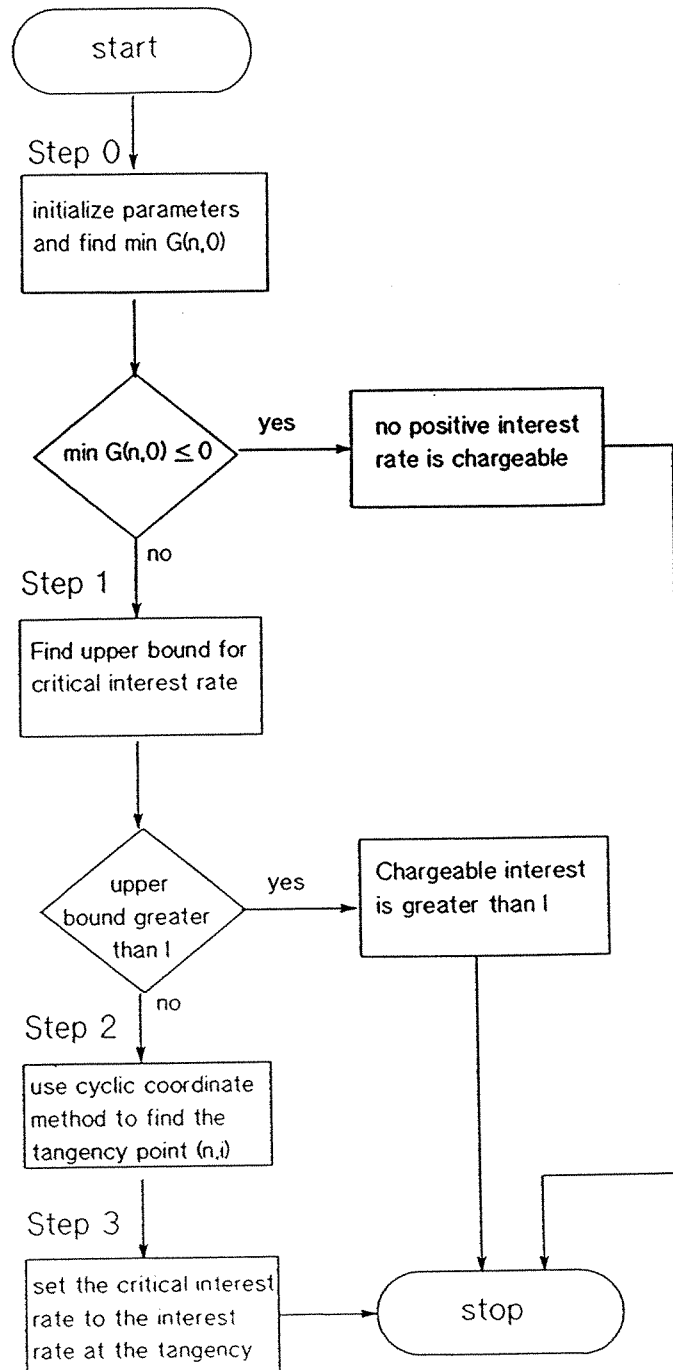


FIGURE A1. Flow chart of solution procedure.

## STEP 1

- 1.1 Set  $i^u \leftarrow i^u + \delta$ .
- 1.2 Find the minimum of  $G(n, i^u)$  using the golden section method. Let the minimum solution be  $(n^u, i^u)$ .
- 1.3 If  $G(n^u, i^u) \geq 0$  and  $i^u \geq I \Rightarrow$  stop and conclude chargeable interest rate is greater than  $I$ .  
If  $G(n^u, i^u) \geq 0$  and  $i^u < I \Rightarrow$  go to step 1.1.  
If  $G(n^u, i^u) < 0 \Rightarrow$  go to step 2.

## STEP 2

- 2.1 Set  $k \leftarrow 1, i^k \leftarrow i^u, n^k \leftarrow n^u$ .
- 2.2 Find  $i^{k+1} \in [0, i^k]$  using the bisection method such that  $G(n^k, i^{k+1}) = 0.0$ .
- 2.3 Find the minimum of  $G(n, i^{k+1})$  using the golden section method. Let the minimum solution be  $(n^{k+1}, i^{k+1})$ .
- 2.4 If  $|| (n^{k+1}, i^{k+1}) - (n^k, i^k) || < \varepsilon$  or  $|| \nabla G(n^k, i^k) || < \varepsilon \Rightarrow$  tangency point found and go to step 3. If  $|| (n^{k+1}, i^{k+1}) - (n^k, i^k) || \geq \varepsilon$  and  $|| \nabla G(n^k, i^k) || \geq \varepsilon \Rightarrow k \leftarrow k + 1$  and go to step 2.2.

## STEP 3

- 3.1 Set  $i^* = i^t = i^{k+1}$  and  $n^t = n^{k+1}$ .

The first step of the algorithm is to initialize the parameters: the error tolerance ( $\varepsilon$ ), interest rate step size ( $\delta$ ), and maximum desirable interest rate ( $I$ ). Here,  $\varepsilon$  should be sufficiently small and positive.  $I$  is typically equal to 1.00 if the period length is annual, and  $\delta$  is restricted to be greater than zero but less than  $I$ . The first test determines if there is any chargeable positive interest rate  $i$  such that  $F_1(n) \geq F_2(n, i)$  for all  $n \in (0, N]$ . This test includes setting  $i = 0$  and searching in the  $n$  direction using the golden search method for the minimum  $G(n, 0)$ . Let the minimum of the function  $G(n, 0)$  be the point  $(n', 0)$ . If  $G(n', 0)$  is less than zero, the solution procedure terminates and concludes there is no positive interest that is chargeable for the given values of  $\rho$ ,  $d$ , and  $N$  (an undesirable situation from the lender's viewpoint). Otherwise, the solution procedure proceeds to Step 1 of the algorithm.

Step 1 calculates an upper bound for  $i^*$ . Let the candidate upper bound interest rate be  $i^u$ . If  $G(n, i^u) < 0$  for any  $n \in (0, N]$ , then  $i^u$  is an upper bound for the critical interest rate  $i^*$ . Let the corresponding period that gives the minimum of the function  $G(n, i^u)$  be  $n^u$ . Hence, this step iteratively updates  $i^u$  by  $\delta$  to find the smallest  $i^u$  such that  $G(n, i^u) < 0$  for any  $n \in (0, N]$ . Once an upper bound interest rate  $i^u$  and its corresponding  $n^u$  is found the solution technique proceeds

to Step 2. If  $i$  has been updated to the value of  $I$  and no upper bound has been found, the solution procedure terminates and concludes the interest rate that is chargeable is greater than  $I$  for the given values of  $\rho$ ,  $d$ , and  $N$  (a desirable situation from the lender's viewpoint).

At Step 2, the cyclic coordinate method is used to find the tangency point  $(n^t, i^t)$  of functions  $F_1(n)$  and  $F_2(n, i)$ . In this step, the upper bound  $i^u$  in Step 1 is iteratively reduced until the tangency point is found. At Step 2.1, the iteration count  $k$  is set to one;  $i^k$  is set to  $i^u$ ; and  $n^k$  is set to  $n^u$ . Since  $G(n^k, 0) > 0$  (Step 0 showed that  $G(n, 0) > 0$  for all  $n \in (0, N]$ ) and  $G(n^k, i^k) < 0$  (from the conclusion of Step 1), there exists an interest rate, say  $i^{k+1} \in [0, i^k]$ , such that  $G(n^k, i^{k+1}) = 0$ . The bisection method is used to search for  $i^{k+1}$ . If  $G(n, i^{k+1}) < 0$  for any  $n \neq n^k$ , then  $i^{k+1}$  is not equal to  $i^*$  and a new upper bound for the tangent interest rate is calculated. Let the minimum of the function  $G(n, i^{k+1})$  be at the point  $(n^{k+1}, i^{k+1})$ . Note that  $G(n^{k+1}, i^{k+1})$  must be less than or equal to  $G(n^k, i^{k+1})$ . Hence,  $G(n^{k+1}, i^{k+1})$  must be less than or equal to zero. If  $G(n^{k+1}, i^{k+1})$  is equal to zero, then  $i^{k+1} = i^*$  and  $n^{k+1} = n^*$ . Thus, a termination test for the solution procedure is to check if the distance between the points  $(n^k, i^k)$  and  $(n^{k+1}, i^{k+1})$  is small (smaller than  $\epsilon$ ). Another termination test is to check if the distance of the gradient function of  $G(n^{k+1}, i^{k+1})$  is smaller than  $\epsilon$  since at the tangency point the distance of the gradient function of  $G(n^t, i^t)$  is zero.

Step 2.4 tests if the solution procedure terminates. If the solution procedure terminates,  $i^*$  and  $i^t$  are set equal to  $i^{k+1}$  and  $n^t$  is set equal to  $n^{k+1}$ . Otherwise, the iteration count  $k$  is updated and the procedure returns to Step 2.2.

If we tried to maximize the function  $f(n, i) = i$  subject to the constraint  $G(n, i) \geq 0$  for all  $n \in (0, N]$  and  $i \in [0, I]$ , the  $\partial f / \partial n$  would equal zero. As a result, the solution depends entirely on the initial value of  $n$  if any algorithm that requires the gradient of  $f(n, i)$  is used. With the objective function minimize  $G(n, i)$ , the  $|\partial G / \partial i|$  is much larger than the  $|\partial G / \partial n|$ . As such, any algorithm using the gradient of  $G(n, i)$  becomes very unreliable. For this reason, a non-gradient based solution procedure is used to determine the tangency point  $(n^t, i^t)$ .

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BIOGRAPHICAL SKETCHES

GERALD A. FLEISCHER is Professor of Industrial and Systems Engineering at the University of Southern California. He received his B.S.I.E from St. Louis University, his M.S.I.E from the University of California, Berkeley, and his Ph.D from Stanford University. A frequent contributor to the literature of engineering economy, Dr. Fleischer's most recent publication is *Introduction to Engineering Economy* published by PWS (1994).

MAGED M. DESSOUKY is Assistant Professor of Industrial and Systems Engineering at the University of Southern California. He received his Ph.D in Industrial Engineering from University of California, Berkeley, M.S.I.E from Purdue University, and B.S.I.E from Purdue University. Prior to joining the faculty at USC, Dr. Dessouky was employed at Hewlett-Packard (Systems Analyst), Bellcore (Member of Technical Staff), and Pritsker Corporation (Senior Systems Analyst).

SHU MING NG is currently a Lecturer of the Department of Management at Hong Kong University of Science and Technology. He received his Ph.D in Industrial Administration from Carnegie Mellon and M.Sc. from Stanford University. Prior to joining HKUST, he was an Assistant Professor of Industrial and Systems Engineering at the University of Southern California.

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