Real-Time Estimation of Travel Times along the Arcs and Arrival Times at the Nodes of Dynamic Stochastic Networks

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***Abstract*— Route planning in uncertain and dynamic networks has recently emerged as an active and intense area of research, both due to industry needs and technological advances. This paper investigates methods to predict travel times along the arcs and estimate arrival times at the nodes of a stochastic and dynamic network in real time. It is shown that, under fairly mild conditions, the developed travel and arrival time estimators are unbiased and that the error variance of the arrival time estimator is bounded. Simulation results are used to demonstrate the efficiency of the proposed algorithm.**

***Index Terms*— Travel time estimator, Arrival time estimator, Dynamic stochastic network, Route planning, Kalman filter.**

I. INTRODUCTION

In many traffic networks, especially in major cities, traffic congestion has already reduced mobility and system reliability. In addition to contributing to drivers' inefficiency, traffic congestion is a major source of air pollution, wasted energy, and increased maintenance cost caused by the volume of vehicles on the roadways. Furthermore, the delay caused by congestion significantly increases the cost of freight movements in the transportation industry and reduce the possibility of just-in-time delivery set by customers (shippers, manufacturers, retailers, etc.). Nowadays, customers are more willing to do business with reliable companies committed to meet their needs.

In the transportation industry, it is widely expected that the deployment of advanced technologies such as the use of information technologies can reduce the level of uncertainties, including delays, to a manageable level. These technologies include: vehicle tracking (such as GPS), wireless communication, navigable map databases, and real-time information services.

Whereas in the past, once the drivers left their origins, it was difficult for them to adjust their routes according to traffic congestion, these technologies make accurate real-time routing a possible reality. The focus of this paper is to improve the routing in uncertain and dynamic environments by developing techniques that can be easily implemented in real-time using new but currently available computer and information technologies.

Recently, route planning in uncertain and dynamic networks has emerged as an active and intense area of research, both due to industry needs and technological advances [1]-[4]. In particular, Kim *et al.* studied optimal vehicle routing in a non-stationary stochastic network [5]. They developed decision making procedures based on a Markov decision process model for determining the optimal driver attendance time, departure times, and routing policies. The methodology was used to develop routing strategies for the stochastic shortest path problem. The authors concluded that real-time traffic information combined with historical data can significantly reduce the expected total costs and vehicle usage.

Ichoua *et al.* investigated the time-dependent vehicle routing problem (TDVRP) with soft time-windows [6]. They presented a time-dependent speed model to calculate the travel times between two nodes. Tabu Search was used to find good routes for the TDVRP problem. Jula *et al.* considered truck route planning for non-stationary stochastic networks with time-windows at customer locations [7]. They developed a methodology to estimate the truck arrival times, and proposed an approximate solution method to find the least-cost route while meeting the required time-windows at the customer locations.

This paper investigates methods to estimate travel times along the arcs and arrival times at the nodes of a dynamic and stochastic network, a step prior to dynamic route planning. Despite its importance and practical application for real-time routing, especially in major cities with traffic congestion, research efforts on developing arrival time estimators have been very limited [8],[9]. On the other hand, real-time travel time estimators have just received attention. In [10] and [11],

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the authors proposed a linear regression method to predict travel times on freeways. Performing numerous experiments on California freeway segments, they observed that there exists a linear relation between the future and instantaneous travel times. More precisely, they have shown that the future travel time on a segment of a highway can be described by a linear model using the instantaneous and historical travel times on that segment. A linear dynamical model was also developed in [12] and [13] to predict travel times on a segment of a highway. The historical data is used to determine a transition function from the current instant of time to the immediate future. The work, however, does not elaborate on how this function can be obtained. Other researches have used Artificial Neural Networks (ANN) [14], [15], and statistical methods [14],[16] extensively to predict travel times on freeway segments.

In this paper, given a route in a dynamic stochastic transportation network, we develop a methodology to estimate arrival times at the nodes of that route. To estimate the arrival times, we first develop a technique to predict the travel times on the arcs of the network by combining historical data with real-time measurements of travel times along the arcs. The technique is developed based on a predictor-corrector form of the Kalman filter in which available historical data are used for predicting the travel times, and real-time measurements are used to correct and update our prediction at each instant of time. It should be noted that, for the sake of simplicity, our travel time predictor ignores possible correlation between adjacent arcs. In other words, to predict the travel time along an arc, we solely use the available historical and real-time data on that arc and ignore possible network affects. It is, however, our intention to expand our travel time prediction model to include the impact of adjacent arcs in our future works.

In summary, our proposed methodology consists of the following two stages:

1. *Predicting travel times on arcs*: Given the time of day together with the historical and real-time data of travel times on the arcs of a transportation network, we predict the future travel times on those arcs, recursively.
2. *Estimating arrival times at nodes*: Given a route in the network, the departure time from the first node of the route, and the predicted travel times on the arcs of the network, the arrival times at the nodes of the route are estimated.

In this paper, we show that, under rather mild assumptions, the developed travel time estimator on the arcs and the arrival time estimator at the nodes are unbiased. We also find an upper bound on the error variance of the arrival time estimator.

The paper is organized as follows. In Section II, using Kalman filtering, a technique is developed to predict the travel times on the arcs of a transportation network. In Section III, an algorithm is developed to estimate the arrival times at the nodes of the network. To evaluate the efficiency of the developed estimators, simulation experiments are generated and discussed in Section IV. Section V concludes this paper.

II. Predicting Travel Times on Arcs

Let $G=(N,A)$ be a transportation network (graph) with node set $N=\{1,\ldots,i,j,\ldots n\}$ and arc set $A=\{(i,j)\,|\,i,j\in N\}$. We assume that the historical data as well as the real-time information of travel times on the arcs of the network are available. Let

T be the length of the planning horizon,

k be an index of time in the planning horizon,

$x_{ij}(k)$ be the travel time between nodes i and j at time k,

$y_{ij}(k)$ be the measured travel time on arc (i,j) at time k,

$x_{ij}^h(k)$ be the historical travel time on arc (i,j) at time k, and

$u_{ij}(k)$ be the historical change in travel time on arc (i,j) from time k to k+1; more precisely,

$$u_{ij}(k)=x_{ij}^h(k+1)-x_{ij}^h(k). \tag{1}$$

As discussed earlier, a linear relationship between the near-future travel time and the instantaneous travel time of a segment of a freeway can be observed [10],[11]. Based on this observation, we propose the linear model

$$\begin{aligned} x_{ij}(k+1) &= x_{ij}(k)+u_{ij}(k)+w_{ij}(k) \\ y_{ij}(k) &= x_{ij}(k)+v_{ij}(k) \end{aligned}, \tag{2}$$

to model the dynamic behavior of travel time on arc (i,j). Later, in Section IV, by means of simulation experiments using real-life data, we will demonstrate the accuracy of the model. In (2), $w_{ij}(k)$ is the travel time disturbance on arc (i,j) at time k and $v_{ij}(k)$ is the error in the travel time measurement at time k, which is caused by sensing devices or human errors. $w_{ij}(k)$ represents the real-time changes in the travel time at time k, which are not included in the historical data $u_{ij}(k)$.

We assume that $u_{ij}(k)$, $w_{ij}(k)$, $v_{ij}(k)$, and $x_{ij}(0)$ are all mutually uncorrelated Gaussian random variables with the following specifications:

$$\begin{aligned} &E\{u_{ij}(k)\}=\eta_{ij}(k); \\ &\qquad E\{u_{ij}(k)u_{ij}(l)\}=\left(\sigma_{ij}^2(k)+\eta_{ij}^2(k)\right)\delta(k-l) \end{aligned}, \tag{3}$$

$$E\{w_{ij}(k)\}=0; \quad E\{w_{ij}(k)w_{ij}(l)\}=q_{ij}^2(k)\delta(k-l), \tag{4}$$

$$E\{v_{ij}(k)\}=0; \quad E\{v_{ij}(k)v_{ij}(l)\}=r_{ij}^2(k)\delta(k-l), \tag{5}$$

$$x_{ij}(0)\ \square\square\ \left(\lambda_{ij}(0),p_{ij}(0)\right). \tag{6}$$

where in (3) to (5), δ function is defined as

$$\delta(\alpha)=\begin{cases}1 & \alpha=0 \\ 0 & Otherwise\end{cases}. \tag{7}$$

In (3), $\eta_{ij}(k)$ and $\sigma_{ij}(k)$ are the mean and standard deviation of $u_{ij}(k)$, respectively. In (4), the mean travel time disturbance on arc (*i,j*) for all times is assumed to be zero, while its variance at time *k* is denoted by $q_{ij}^2(k)$. Similarly, in (5) the mean error of the travel time measurement for all times is assumed to be zero. The variance at time *k* is given by $r_{ij}^2(k)$. In (6), we assumed that the mean and variance of the travel time between nodes *i* and *j* at time *k*=0 (the initial time) is $\lambda_{ij}(0)$ and $p_{ij}(0)$, respectively.

A. Linear Model Parameters

In this subsection, we will briefly discuss some ways to determine the values of our linear model parameters. In particular, we explain how to determine $r_{ij}^2(k)$, $\eta_{ij}^2(k)$, $\sigma_{ij}^2(k)$, and $q_{ij}^2(k)$ in (3) to (5).

To determine $r_{ij}^2(k)$, we know that $v_{ij}(k)$ is the travel time measurement error. The variance $r_{ij}^2(k)$ can be calculated off-line by setting up a set of measurements to determine the difference between the actual and measured travel times on arc (*i,j*) at time *k*. For instance, if the travel times are measured using magnetic loops, probe vehicles can de used to determine the actual travel times.

To determine $\eta_{ij}^2(k)$ and $\sigma_{ij}^2(k)$, we notice that in (1) $u_{ij}(k)$, the "actual" historical change in travel time on arc (*i,j*), is developed from $x_{ij}^h(k)$. In reality, according to (2), the "actual" historical travel time on arc (*i,j*), $x_{ij}^h(k)$, may not be available due to the measurement error $v_{ij}(k)$. What is available is the "observed" historical travel time, which we denote it by $y_{ij}^h(k)$. That is, although the model in (2) is based on the "actual" historical change in travel time, $u_{ij}(k)=x_{ij}^h(k+1)-x_{ij}^h(k)$, the available data is the "observed" historical change in travel time, $\hat{u}_{ij}(k)=y_{ij}^h(k+1)-y_{ij}^h(k)$. Assuming that $x_{ij}^h(k)$ and $v_{ij}(k)$ are mutually uncorrelated and $E\{v_{ij}(k)\}=0$, one can easily show that

$$\begin{aligned}E\{\hat{u}_{ij}(k)\}&=E\{y_{ij}^h(k+1)-y_{ij}^h(k)\}\\&=E\{x_{ij}^h(k+1)+v_{ij}(k+1)-x_{ij}^h(k)-v_{ij}(k)\}\\&=E\{x_{ij}^h(k+1)-x_{ij}^h(k)\}\\&=E\{u_{ij}(k)\}=\eta_{ij}(k)\end{aligned}. \quad (8)$$

That is the mean "observed" historical change and the mean "actual" historical change are equal. In addition,

$$\begin{aligned}&E\{\hat{u}_{ij}(k)\hat{u}_{ij}(l)\}\\&=E\{(y_{ij}^h(k+1)-y_{ij}^h(k))(y_{ij}^h(l+1)-y_{ij}^h(l))\}\end{aligned}. \quad (9)$$

$$\begin{aligned}&=E\{u_{ij}(k)u_{ij}(l)\}+E\{v_{ij}(k+1)v_{ij}(l+1)\}\\&\quad+E\{v_{ij}(k)v_{ij}(l)\}-E\{v_{ij}(k)v_{ij}(l+1)\}-E\{v_{ij}(k+1)v_{ij}(l)\}\end{aligned}$$

Let *k*=*l* in (9). According to (5), we have

$$E\{\hat{u}_{ij}(k)\hat{u}_{ij}(k)\}=E\{u_{ij}(k)u_{ij}(k)\}+r_{ij}^2(k+1)+r_{ij}^2(k). \quad (10)$$

Subtracting $\eta_{ij}^2(k)$ from both sides and using (8) we can simplify (10) as

$$\hat{\sigma}_{ij}^2(k)=\sigma_{ij}^2(k)+r_{ij}^2(k+1)+r_{ij}^2(k). \quad (11)$$

Equation (11) presents the relationship between the variance of the "observed" historical change, $\hat{\sigma}_{ij}^2(k)$, and the variance of the "actual" historical change, $\sigma_{ij}^2(k)$. Thus, according to (8) and (11) the statistical mean and variance of the observed historical change in travel time can be used to determine the mean and variance of the actual historical change in travel time. As we will see later, this is sufficient information for our travel time prediction.

Finally, to determine $q_{ij}^2(k)$, we run the linear model many times and record the difference between the predicted and actual travel times (which can be obtained using probe vehicles). That information is used to calculate $q_{ij}^2(k)$.

In the following, we use the corrector-predictor form of the Kalman-Filter to estimate the future travel times on the arcs of a graph *G*.

B. Single-Stage Predictor

In the single stage predictor, given the measured travel time $y_{ij}(k)$ on arc (*i,j*) at time *k*, we estimate the travel time $x_{ij}(k+1)$ on the arc at time *k*+1. We know that the estimator that minimizes the mean-squared error of the estimation error is given by [17]

$$\hat{x}_{ij}(k+1|k)=E\{x_{ij}(k+1)|y_{ij}(k)\}, \quad (12)$$

where $\hat{x}_{ij}(k+1|k)$ denotes the estimate of travel time $x_{ij}(k+1)$ given measurement $y_{ij}(k)$, and is called the mean-squared *predicted estimator* of $x_{ij}(k+1)$. Using our dynamical model in (2), the predicted estimator $\hat{x}_{ij}(k+1|k)$ can be written as

$$\begin{aligned}\hat{x}_{ij}(k+1|k)&=E\{x_{ij}(k)+u_{ij}(k)+w_{ij}(k)|y_{ij}(k)\}\\&=\hat{x}_{ij}(k|k)+\eta_{ij}(k)\end{aligned}, \quad (13)$$

where $\hat{x}_{ij}(k|k)$ is the mean-squared *filtered estimator* of $x_{ij}(k)$ given $y_{ij}(k)$. Equation (13) indicates that to calculate the predicted estimator $\hat{x}_{ij}(k+1|k)$ the value of the filtered estimator $\hat{x}_{ij}(k|k)$ should first be obtained. In Section II.D, we will see how to determine the filtered estimator $\hat{x}_{ij}(k|k)$.

Let us denote by $\tilde{x}_{ij}(k+1|k)$ the single-stage *predictor error*, which is defined as

$$\tilde{x}_{ij}\left(k+1|k\right)=x_{ij}\left(k+1\right)-\hat{x}_{ij}\left(k+1|k\right). \quad (14)$$

Note that since $x_{ij}\left(k+1\right)$ and $y_{ij}\left(k\right)$ are joint Gaussian random variables, the estimator in (12) is unbiased [17]; hence, we have

$$E\left\{\tilde{x}_{ij}\left(k+1|k\right)\right\}=0. \quad (15)$$

Using (15), the error variance of the estimator is given by

$$\begin{aligned}\operatorname{var}\left(\tilde{x}_{ij}\left(k+1|k\right)\right)&=E\left\{\tilde{x}_{ij}^{2}\left(k+1|k\right)\right\}\\&=E\left\{\left(\tilde{x}_{ij}\left(k|k\right)+u_{ij}\left(k\right)-\eta_{ij}\left(k\right)+w_{ij}\left(k\right)\right)^{2}\right\},\end{aligned} \quad (16)$$

where $\tilde{x}_{ij}\left(k|k\right)=x_{ij}\left(k\right)-\hat{x}_{ij}\left(k|k\right)$ is the filter error which is the error of estimating $x_{ij}\left(k\right)$ given $y_{ij}\left(k\right)$. Since $\tilde{x}_{ij}\left(k|k\right)$, $\left(u_{ij}\left(k\right)-\eta_{ij}\left(k\right)\right)$, and $w_{ij}\left(k\right)$ are statistically uncorrelated, (16) can be simplified as

$$\begin{aligned}\operatorname{var}\left(\tilde{x}_{ij}\left(k+1|k\right)\right)&=E\left\{\tilde{x}_{ij}^{2}\left(k|k\right)\right\}+\sigma_{ij}^{2}\left(k\right)+q_{ij}^{2}\left(k\right)\\&=\operatorname{var}\left(\tilde{x}_{ij}\left(k|k\right)\right)+\sigma_{ij}^{2}\left(k\right)+q_{ij}^{2}\left(k\right)\end{aligned}, \quad (17)$$

where in deriving (17), we use the fact that the filter error is unbiased [17], i.e.

$$E\left(\tilde{x}_{ij}\left(k|k\right)\right)=0. \quad (18)$$

Equations (15) and (17) provide the mean and variance of the single-stage *predictor error* of the mean-squared *predicted estimator* of $x_{ij}\left(k+1\right)$ given in (13).

C. *m*th-Stage Predictor

Here, we extend the single-stage predictor to the *m*th-stage predictor, such that given the measured travel time $y_{ij}\left(k\right)$ at time *k* on arc (*i*,*j*), we determine an unbiased estimate of travel time on that arc at time *k*+*m*, where *m*≥1. Similar to the single-stage predictor, we know that the estimator that minimizes the mean-squared estimation error is given by

$$\hat{x}_{ij}\left(k+m|k\right)=E\left\{x_{ij}\left(k+m\right)|y_{ij}\left(k\right)\right\}, \quad (19)$$

where $\hat{x}_{ij}\left(k+m|k\right)$ is the estimate of $x_{ij}\left(k+m\right)$ given the measurement $y_{ij}\left(k\right)$. Using (2) recursively, we have

$$\begin{aligned}&\hat{x}_{ij}\left(k+m|k\right)\\&=E\left\{x_{ij}\left(k\right)+\sum_{l=0}^{m-1}u_{ij}\left(k+l\right)+\sum_{l=0}^{m-1}w_{ij}\left(k+l\right)|y_{ij}\left(k\right)\right\}. \quad (20)\\&=\hat{x}_{ij}\left(k|k\right)+\sum_{l=0}^{m-1}\eta_{ij}\left(k+l\right)\end{aligned}$$

The *m*th-stage predictor error is defined as $\tilde{x}_{ij}\left(k+m|k\right)=x_{ij}\left(k+m\right)-\hat{x}_{ij}\left(k+m|k\right)$. Similar to the single stage predictor, the estimator in (20) is unbiased since $x_{ij}\left(k+m\right)$ and $y_{ij}\left(k\right)$ are joint Gaussian random variables for all *m*≥1, i.e.,

$$E\left\{\tilde{x}_{ij}\left(k+m|k\right)\right\}=0. \quad (21)$$

The error variance of the *m*th-stage predictor is

$$\begin{aligned}\operatorname{var}\left(\tilde{x}_{ij}\left(k+m|k\right)\right)&=E\left\{\tilde{x}_{ij}^{2}\left(k+m|k\right)\right\}. \quad (22)\\&=E\left\{\left(\tilde{x}_{ij}\left(k|k\right)+\sum_{l=0}^{m-1}\left(u_{ij}\left(k+l\right)-\eta_{ij}\left(k+l\right)\right)+\sum_{l=0}^{m-1}w_{ij}\left(k+l\right)\right)^{2}\right\}\end{aligned}$$

Using (3) and (4), we know that $\tilde{x}_{ij}\left(k|k\right)$, $\left(u_{ij}\left(k+l\right)-\eta_{ij}\left(k+l\right)\right)$, and $w_{ij}\left(k+l\right)$ are statistically uncorrelated for $\forall l=0,...,m-1$, therefore

$$\begin{aligned}&\operatorname{var}\left(\tilde{x}_{ij}\left(k+m|k\right)\right)\\&=\operatorname{var}\left(\tilde{x}_{ij}\left(k|k\right)\right)+\sum_{l=0}^{m-1}\sigma_{ij}^{2}\left(k+l\right)+\sum_{l=0}^{m-1}q_{ij}^{2}\left(k+l\right),\end{aligned} \quad (23)$$

where in deriving (23), we use (18). Equations (21) and (23) present the mean and variance of the *m*th-stage *predictor error* of the mean-squared *predicted estimator* of $x_{ij}\left(k+m\right)$ given in (20). Equation (23) indicates that as *m* (the number of stages of prediction) increases, the variance of the prediction error increases too. In other words, as *m* becomes larger, there exists more uncertainties in estimating travel time on arc (*i*,*j*) using the current information at time *k*.

D. *State Filter*

As seen from (13) and (20), the predicted estimate of the travel time $x_{ij}\left(k+m\right)$ on arc (*i*,*j*) at time *k*+*m*, *m*≥1, depends on the value of the filtered estimator $\hat{x}_{ij}\left(k|k\right)$ (i.e., the estimate of travel time $x_{ij}\left(k\right)$ at time *k*) given the measured travel time $y_{ij}\left(k\right)$ at time *k*. The predictor-corrector form of the Kalman filter is used here to calculate the filtered estimator $\hat{x}_{ij}\left(k|k\right)$ as follows [17]

$$\hat{x}_{ij}\left(k|k\right)=\hat{x}_{ij}\left(k|k-1\right)+\mathrm{K}_{ij}\left(k\right)\tilde{y}_{ij}\left(k|k-1\right), \quad (24)$$

where for the dynamic model in (2), $\tilde{y}_{ij}\left(k|k-1\right)$ is the *measurement residual process* of $y_{ij}\left(k\right)$, and is defined as

$$\tilde{y}_{ij}\left(k|k-1\right)=y_{ij}\left(k\right)-\hat{x}_{ij}\left(k|k-1\right). \quad (25)$$

$\mathrm{K}_{ij}\left(k\right)$ is the *Kalman gain* of arc (*i*,*j*) at time *k* which is specified by the following set of equations:

$$\begin{aligned}\mathrm{K}_{ij}\left(k\right)&=\frac{\operatorname{cov}\left(x_{ij}\left(k\right),\tilde{y}_{ij}\left(k|k-1\right)\right)}{\operatorname{var}\left(\tilde{y}_{ij}\left(k|k-1\right)\right)}\\&=\frac{\operatorname{var}\left(\tilde{x}_{ij}\left(k|k-1\right)\right)}{\operatorname{var}\left(\tilde{x}_{ij}\left(k|k-1\right)\right)+r_{ij}^{2}(k)}\end{aligned}, \quad (26)$$

and from (17)

$$\begin{aligned}\operatorname{var}\left(\tilde{x}_{ij}\left(k|k-1\right)\right) \\ = \operatorname{var}\left(\tilde{x}_{ij}\left(k-1|k-1\right)\right)+\sigma_{ij}^2\left(k-1\right)+q_{ij}^2\left(k-1\right)\end{aligned}, \quad (27)$$

and

$$\operatorname{var}\left(\tilde{x}_{ij}\left(k|k\right)\right)=\left(1-\mathrm{K}_{ij}\left(k\right)\right)\operatorname{var}\left(\tilde{x}_{ij}\left(k|k-1\right)\right). \quad (28)$$

It is always possible that the value of $\operatorname{var}\left(\tilde{x}_{ij}\left(k|k-1\right)\right)$ be zero for some *k*, which makes (26) ill-defined if $r_{ij}\left(k\right)=0$. Therefore, we require that $r_{ij}\left(k\right)>0$ for all *k*.

Note that the error variance of travel time on arc (*i*,*j*) at time *k*=0 is given by (6), i.e., $\operatorname{var}\left(\tilde{x}_{ij}\left(0|0\right)\right)=p_{ij}\left(0\right)$, which initializes (26) to (28). In other words, given $\operatorname{var}\left(\tilde{x}_{ij}\left(0|0\right)\right)$, $\sigma_{ij}^2\left(k\right)$, $q_{ij}^2\left(k\right)$, and $r_{ij}^2\left(k\right)$, $\forall k=0,1,2,\ldots$ by (3) to (6), all variables in (26) to (28) can be determined recursively in the following order:

$$\begin{aligned}&\operatorname{var}\left(\tilde{x}_{ij}\left(0|0\right)\right)\xrightarrow{\text{using (27)}}\operatorname{var}\left(\tilde{x}_{ij}\left(1|0\right)\right)\\ &\xrightarrow{\text{using (26)}}\mathrm{K}_{ij}\left(1\right)\xrightarrow{\text{using (28)}}\operatorname{var}\left(\tilde{x}_{ij}\left(1|1\right)\right)\rightarrow\cdots\end{aligned}. \quad (29)$$

In particular, the Kalman gain $\mathrm{K}_{ij}\left(k\right)$ of arc (*i*,*j*) at each time *k* is determined by (29).

Note also that (24) is the predictor-corrector form of the Kalman filter, in which the predictor step $\hat{x}_{ij}\left(k|k-1\right)$ uses information from (13) to predict the next step, and the corrector step $\mathrm{K}_{ij}\left(k\right)\tilde{y}_{ij}\left(k|k-1\right)$ uses the new measurement to update or correct the predictor estimation $\hat{x}_{ij}\left(k|k-1\right)$.

The filtered estimate of travel time on arc (*i*,*j*) at time *k*=0 is also given by (6), i.e., $\hat{x}_{ij}\left(0|0\right)=\lambda_{ij}\left(0\right)$, which initializes equations (13) and (24). In other words, given $\hat{x}_{ij}\left(0|0\right)$, $\eta_{ij}\left(k\right)$, and $\mathrm{K}_{ij}\left(k+1\right)$, $\forall k=0,1,2,\ldots$ by (6), (3), and (29), respectively, together with the measured travel time $y_{ij}\left(k\right)$, the mean-squared *predicted estimate* and mean-squared *filtered estimate* of $x_{ij}\left(k\right)$ can be calculated recursively as follows:

$$\hat{x}_{ij}\left(0|0\right)\xrightarrow{\text{using (18)}}\hat{x}_{ij}\left(1|0\right)\xrightarrow{\text{using (24)}}\hat{x}_{ij}\left(1|1\right)\rightarrow\cdots. \quad (30)$$

Therefore, at each time *k* and by using the *m*th-stage predictor in (20), the travel time $\hat{x}_{ij}\left(k+m|k\right)$ on arc (*i*,*j*) at time *k*+*m* can be predicted.

III. Estimating Arrival Times at Nodes

In the previous section, we developed a technique to predict the travel times on the arcs of a given transportation network. In this section, we use the predicted travel times on the arcs to estimate the arrival times at the nodes of the network. We show that under rather mild assumptions the developed arrival time estimator is unbiased. In this section, we also find a bound on the error variance of the estimator.

We define route *r* in graph *G* as a sequence of nodes visited in the specified order. Figure 1 shows a typical route *r*, which for convenience is represented as an ordered set, i.e., $r=\{1,2,\ldots,i,j,\ldots,d\}$. Let also A^r be the arc set associated with route *r* which is defined as

$$A^r=\left\{(i,j)\,|\,i,j\in r,\text{and } j \text{ is visited immidiately after } i\right\}. \quad (31)$$

{Figure 1 to be added here.}

We assume that the departure time from node 1 on route *r*, denoted by z_1^r, is given. We also assume that, using the developed technique in Section II, the predicted travel times on the arcs of graph *G* are available. Given z_1^r, the arrival times at all the nodes of route *r* can be determined using the following set of equations,

$$\begin{aligned}z_2^r&=z_1^r+x_{12}\left(z_1^r\right)\\ z_3^r&=z_2^r+x_{23}\left(z_2^r\right)\\ &\vdots\\ z_j^r&=z_i^r+x_{ij}\left(z_i^r\right)\\ &\vdots\\ z_d^r&=z_{d-1}^r+x_{d-1,d}\left(z_{d-1}^r\right)\end{aligned}, \quad (32)$$

where z_i^r is the arrival time at node *i* taking route *r*, and $x_{ij}\left(z_i^r\right)$ is the travel time on arc (*i*,*j*)$\in A^r$ at time z_i^r. Note that in (32) we assumed that the departure time from node *i* is equal to the arrival time at that node. This is achieved by assuming that there is no service time at the nodes of the network. Note also that, in general, z_i^r takes a real value, whereas the methodology developed in Section II is based on integer numbers of discrete-time indices *k*'s. That is, the argument of x_{ij} should be an integer number.

Generally speaking, the dynamics of many transportation networks are slow. Therefore, by selecting an appropriate sampling time T_S, the error of approximating x_{ij} at time z_i^r with its value at the nearest sampling time would be negligible. More precisely, let $k\cdot T_S\le z_i^r<\left(k+1\right)\cdot T_S$, where *k* is an integer number representing the time index. With an appropriate selection of T_S, the error of approximating x_{ij} at time z_i^r with $x_{ij}\left(k\cdot T_s\right)$, which for the sake of simplicity has been denoted by $x_{ij}\left(k\right)$, is negligible.

Given the sampling period T_S and the arrival time z_i^r, we define and use the *discretizing* function $\Theta:\square\rightarrow\square$, which returns the discrete-time index, as

$$\Theta(z_i) = \left\lfloor \frac{z_i}{T_S} \right\rfloor, \quad (33)$$

where the *floor function* $\lfloor \cdot \rfloor$ returns the largest integer equal to or less than its argument.

A. Estimating the arrival time at the second node

In this subsection, we estimate the arrival time at the second node of route r. Given z_1^r and using (32), the mean-squared estimate of the arrival time at node 2 is

$$\begin{aligned} \hat{z}_2^r &= E\left\{z_2^r \middle| z_1^r\right\} = E\left\{z_1^r + x_{12} \middle| z_1^r\right\} = z_1^r + E\left\{x_{12} \middle| z_1^r\right\} \\ &\approx \hat{z}_1^r + \hat{x}_{12}\left(\Theta(z_1^r) \middle| \Theta(z_1^r)\right) \end{aligned}, \quad (34)$$

where Θ is the discretizing function and $\hat{x}_{12}\left(\Theta(z_1^r) \middle| \Theta(z_1^r)\right)$ is the mean-squared *filtered estimate* of travel time on arc (1,2) at time $\Theta(z_1^r)$ given the measured travel time $y_{12}\left(\Theta(z_1^r)\right)$ at time $\Theta(z_1^r)$.

We denote by $\tilde{z}_2^r$ the error of the estimator in (34), which is defined as $\tilde{z}_2^r = z_2^r - \hat{z}_2^r$. Recall from Section II that $\hat{x}_{12}(k|k)$ $\forall k = 0,1,2,\ldots$ has a normal distribution. $\hat{x}_{12}(k|k)$ and z_1^r are joint Gaussian random variables. Hence, the estimator in (34) is unbiased, and the error variance of the estimator is

$$\begin{aligned} &\operatorname{var}(\tilde{z}_2^r) = E\left\{\left(z_2^r - \hat{z}_2^r\right)^2\right\}. \quad (35) \\ &\approx E\left\{\left(x_{12} - \hat{x}_{12}\left(\Theta(z_1^r) \middle| \Theta(z_1^r)\right)\right)^2\right\} = \operatorname{var}\left(\tilde{x}_{12}\left(\Theta(z_1^r) \middle| \Theta(z_1^r)\right)\right) \end{aligned}$$

B. Estimating the arrival time at the other nodes

Likewise, given z_1^r, the mean-squared estimate of the arrival time at node 3 is

$$\begin{aligned} \hat{z}_3^r &= E\left\{z_3^r \middle| z_1^r\right\} = E\left\{z_2^r + x_{23} \middle| z_1^r\right\} = E\left\{z_2^r \middle| z_1^r\right\} + E\left\{x_{23} \middle| z_1^r\right\} \\ &\approx \hat{z}_2^r + \hat{x}_{23}\left(\Theta(z_2^r) \middle| \Theta(z_1^r)\right) \end{aligned}, \quad (36)$$

where $\hat{x}_{23}\left(\Theta(z_2^r) \middle| \Theta(z_1^r)\right)$ is the mean-squared *predicted estimate* of x_{23} at time $\Theta(z_2^r)$ given the measured travel time $y_{23}\left(\Theta(z_1^r)\right)$ at time $\Theta(z_1^r)$.

It should be noted that the estimator in (36) cannot be realized since the value of z_2^r is not available. To overcome this problem, in (36) we use the estimate $\hat{z}_2^r$, computed in (34), instead of z_2^r. Therefore, the estimator in (36) can be approximated by

$$\hat{z}_3^r \approx \hat{z}_2^r + \hat{x}_{23}\left(\Theta(\hat{z}_2^r) \middle| \Theta(z_1^r)\right). \quad (37)$$

Later in this paper, we show that with appropriate selection of sampling period T_S, the error of estimation in (37) is negligible. Moreover, we find a bound on the error variance of the estimator in (37).

Likewise, assuming that node i precedes node j on route r and that $\hat{z}_i^r$, the arrival time estimate at node i given z_1^r, is given, $\hat{z}_i^r$, the arrival time at node j on route r, can be estimated by

$$\hat{z}_j^r = E\left\{z_j^r \middle| z_1^r\right\} \approx \hat{z}_i^r + \hat{x}_{ij}\left(\Theta(\hat{z}_i^r) \middle| \Theta(z_1^r)\right). \quad (38)$$

In the following we investigate the conditions under which the estimator in (38) will be unbiased.

Definition 1 (Bounded function) *The discrete-time function* $\mu : \square \to \square$ *is said to be bounded in time if there exists a* $\bar{\mu} < \infty$ *such that* $|\mu(k)| \le \bar{\mu}$, $\forall k = 0,1,2,\ldots$.

Assuming that $E\{u_{ij}(k)\} = \eta_{ij}(k)$ in (3) is bounded by $\bar{\eta}_{ij} < \infty$, we have the following theorem:

Theorem 1 (unbiased estimator) *If* $\hat{z}_i^r$ *is an unbiased estimator of* z_i^r *and* $T_S \to 0$, *then the arrival time estimator* $\hat{z}_j^r$ *in (38) will be an unbiased estimator of* z_j^r.

Proof: See Appendix.

We denote by $\tilde{z}_j^r$ the error of estimation in (38) which is defined as $\tilde{z}_j^r = z_j^r - \hat{z}_j^r$. The following theorem gives a bound on the error variance of arrival time estimator $\hat{z}_j^r$ in (38).

Theorem 2 (Error variance bound) *If* $\hat{z}_i^r$ *is an unbiased estimator of* z_i^r *with error variance equal to* $\operatorname{var}(\tilde{z}_i^r)$ *and* $T_S \to 0$, *then the error variance of estimator* $\hat{z}_j^r$ *in (38) is bounded by*

$$\begin{aligned} \operatorname{var}(\tilde{z}_j^r) \le &\left(1 + 2\bar{\eta}_{ij}\right)\operatorname{var}(\tilde{z}_i^r) + \operatorname{var}\left(\tilde{x}_{ij}\left(\Theta(z_1^r) \middle| \Theta(z_1^r)\right)\right) \\ &+ \sum_{k=\Theta(z_1^r)}^{\Theta(\hat{z}_i^r)-1} \sigma_{ij}^2(k) + \sum_{k=\Theta(z_1^r)}^{\Theta(\hat{z}_i^r)-1} q_{ij}^2(k) \end{aligned}. \quad (39)$$

Proof: See Appendix.

In summary, given z_1^r, the departure time from node 1, the arrival time at each node i on route r can be estimated using the developed arrival time estimator in (38) and the predicted travel times on arcs in (20), recursively, in the following order:

$$\begin{aligned} &z_1^r, x_{12}^r\left(\Theta(z_1^r) \middle| \Theta(z_1^r)\right) \xrightarrow{\text{using (38)}} \hat{z}_2^r \\ &\xrightarrow{\text{using (20)}} x_{23}^r\left(\Theta(\hat{z}_2^r) \middle| \Theta(z_1^r)\right) \to \cdots \end{aligned}. \quad (40)$$

We have shown in Theorems 1 and 2 that by properly selecting the sampling period T_S, the arrival time estimator in (38) is an unbiased estimator, and its error variance is bounded by (39). Note that by using Equations (23) and (39), the bound on the error variance of the arrival time at each node j on route r can be written as

$$\operatorname{var}\left(\tilde{z}_j^r\right) \leq \left(1+2\overline{\eta}_{ij}\right)\operatorname{var}\left(\tilde{z}_i^r\right)+\operatorname{var}\left(\tilde{x}_{ij}\left(\Theta\left(\hat{z}_i^r\right)\middle|\Theta\left(z_1^r\right)\right)\right) \quad (41)$$

Note also that given z_1^r, the error variance of the departure time from node 1 is zero, i.e., $\operatorname{var}\left(\tilde{z}_1^r\right)=0$, which together with the given $\operatorname{var}\left(\tilde{x}_{ij}\left(0|0\right)\right)=p_{ij}\left(0\right)$ initializes equations (23), (28), and (41). Hence, the bound on the error variance of the arrival time at each node j on route r can be calculated, recursively.

As seen from (41), the bound on the error variance of the arrival time at each node j, $\operatorname{var}\left(\tilde{z}_j^r\right)$, depends on two elements: 1) the error variance of the arrival time estimator at the previous node, $\operatorname{var}\left(\tilde{z}_i^r\right)$, and 2) the error variance of the travel time predictor on arc (i,j) at time $\hat{z}_i^r$, i.e, $\operatorname{var}\left(\tilde{x}_{ij}\left(\Theta\left(\hat{z}_i^r\right)\middle|\Theta\left(z_1^r\right)\right)\right)$. This fact indicates that the error variance increases sharply as we estimate the arrival time at the nodes further down on the route. It also implies that as we predict the travel time in the distant future, the error variance in (41) increases abruptly.

IV. Simulation Experiments

In this section, we perform simulation experiments to evaluate the efficiency of the developed algorithms in sections II and III.

A. Simulation Experiment 1 (Travel time estimator)

In this simulation experiment, we evaluate the dynamic model in (2) using real world data extracted from the Freeway Performance Measurement System (PeMS). PeMS is an online repository of traffic data for California highways run by the University of California at Berkley in cooperation with the California Department of Transportation, California Partners for Advanced Transit and Highways, and Berkeley Transportation [19]. Data used for testing is obtained from 141 single-loop detectors along the segment called Route D07 East bound, between Los Angeles and West Covina, in California, USA. This segment is about 13.8 miles long starting from highway 101-South/Aliso St. and ending at highway 10-East/Vincent Ave. The travel time measurements were done every 5 minutes for 10 consecutive Tuesdays between January 31 and April 4, 2006. Figure 2 shows the travel time along the segment versus the time of day for three of those 10 Tuesdays.

{*Figure 2 to be added here.*}

Figure 2 implies that the travel time along the segment is very uncertain (even for the same day of the week – here Tuesdays) and that it has a distinctive afternoon congestion around 6:00 p.m.

The measured travel times along the segment on different dates are used to find the values of $\hat{u}_{ij}\left(k\right)=y_{ij}^h\left(k+1\right)-y_{ij}^h\left(k\right)$ for every Tuesday and for every instant of time, k. (See Subsection II.A for more detail.) The values of $\eta_{ij}\left(k\right)$ and $\sigma_{ij}\left(k\right)$ in (3) are then approximated through calculating the statistical mean and standard deviation of observed historical change in travel time, $\hat{u}_{ij}\left(k\right)$, for each k over the 10 selected Tuesdays.

Given the historical (i.e., data from Tuesdays January 31 till April 4, 2006) and real-time (i.e., every five minutes travel time along the segment) data, the developed algorithm in Section II is used to design a single-stage travel time predictor for Tuesdays April 11, 2006 and April 18, 2006. That is, given the historical and real-time data, we predict the travel time along the segment for the next 5 minutes for April 11 and April 18, 2006. Figure 3 and Figure 4 show the actual travel times (solid lines) as well as the predicted ones (dotted lines) versus the time of day. In our models, we assume that the values of $q_{ij}\left(k\right)$, and $r_{ij}\left(k\right)$ in (4) and (5) are 0.1 and 0.05 minutes, respectively, for all values of k.

{Figure 3 *to be added here.*}

{Figure 4 *to be added here.*}

A seen from Figure 3 and Figure 4, our estimators were very efficient in predicting travel times for the next 5 minutes. Note that, although the traffic on Tuesday April 11 follows almost the same pattern as other Tuesdays in our historical, the traffic on April 18, 2006 shows a somewhat different one. In addition to the anticipated afternoon congestion, we can easily observe two minor instances of congestion, one in the morning around 8 a.m. and one in the evening around 10 pm.

To better observe and evaluate the performance of the predictor in case of anticipated and unanticipated congestion, the portion of Figure 4 located between 2:00 pm and 10:00 pm is zoomed in and shown in Figure 5. The zoomed-in area consists of two instances of congestion. The first peak, which is occurred around 4:00 pm, is anticipated congestion (with regard to the historical data illustrated in Figure 2), while the second peak (around 10:00 pm) is an unanticipated one. As seen from Figure 5 there are minor lags between the predicted and actual travel times. The lag is more pronounced around the second peak compared to the first one. While the phenomenon is interesting, it is not surprising. Note that due to the term $u_{ij}(t)$ in (2), the developed predictor expects an instance of congestion around 4:00 pm, thus, it predicts the future travel time around 4:00 pm with less error. The second instance of congestion is an unexpected one. The predictor needs to obtain the real-time measurements first, and then adapts itself to the new traffic pattern. However, no matter if the daily travel time follows a new pattern, the travel time predictor was able to predict the travel times with fairly small amount of error.

{*Figure 5 to be added here.*}

We also designed a 6th-stage predictor for the segment of the highway discussed above. The 6th-stage predictor predicts the travel time along the segment for the next 30 minutes based on current travel time and available historical data. The result is shown in Figure 6.

{*Figure 6 to be added here.*}

Figure 6 illustrates that although the travel time prediction is done for the distant future (here 30 minutes), our predictor is still fairly efficient. As expected and seen from Figure 6, there exist more uncertainties in estimating the travel times further in the future using the current information. Moreover, the lags between the predicted and actual travel times, in particular when the unanticipated congestion occurred, are more noticeable.

B. Simulation Experiment 2 (Stationary stochastic route)

In this simulation experiment, we consider route *r* as shown in Figure 7 which consists of 8 nodes. We assume that route *r* is a stationary stochastic route. The numbers adjacent to the solid lines in Figure 7 are the mean travel times between each two adjacent nodes; for instance, the mean travel time between nodes 2 and 3 is 26 minutes.

{*Figure 7 to be added here.*}

Since route *r* is stationary, the route characteristics, given by Equations (3) to (5), are independent of time *k*. Here, we assume that the network parameters (3) to (5) are equal for all arcs (*i*,*j*), and, hence, the network characteristics are given by

$$E\left\{u_{ij}(k)\right\}=0; \quad E\left\{u_{ij}(k)u_{ij}(l)\right\}=\sigma^2\delta(k-l), \tag{42}$$

$$E\left\{w_{ij}(k)\right\}=0; \quad E\left\{w_{ij}(k)w_{ij}(l)\right\}=q^2\delta(k-l), \tag{43}$$

$$E\left\{v_{ij}(k)\right\}=0; \quad E\left\{v_{ij}(k)v_{ij}(l)\right\}=r^2\delta(k-l). \tag{44}$$

Note that (42) and (43) together with the dynamic model in (2) necessitate that the mean travel time between any two nodes i and j be constant in time equal to its initial value $x_{ij}(0)$, i.e.,

$$E\left\{x_{ij}(k)\right\}=x_{ij}(0) \quad \forall k=1,2,\ldots \tag{45}$$

Thus, the numbers in Figure 7 are, in fact, the initial values (i.e., at time 0) of travel times along the corresponding arcs.

Table 1 column 4 shows the estimated arrival times at the nodes of route r for different values of σ, q, and γ, respectively. Column 5 presents an *approximate* upper bound of the error variance of the estimation at the nodes of route *r* using (41). It is an approximate upper bound since (41) results in a true upper bound of the error variance only when $T_S \to 0$.

To evaluate the arrival time estimator, route *r* in Figure 7 is simulated 1000 times using a Guassian random number generator with parameters given by (42) to (45). The last two columns in Table 1 show the results of the simulated route. Column 6 presents the mean arrival times and Column 7 shows the error variance of the arrival times at the nodes of the simulated route based on the results of the 1000 trials. In this simulation experiment, we assume that $T_S = 5$ minutes. We also assume that the departure time from node 1 on route *r* is 0:10′. That is the algorithm is used after observing the dynamics of the network for 10 minutes to estimate the arrival time at the other nodes. During these 10 minutes, the algorithm initializes and updates its parameters, in particular the Kalman gain, in real-time according to the route dynamics.

{*Table 1 to be added here.*}

Table 1 indicates that the developed algorithm was very efficient in estimating the arrival times at the nodes of route *r*. As expected and seen from Table 1, the error of the arrival time estimator (i.e., the difference between columns 4 and 6 in Table 1) increases for the larger values of σ, q, and γ, and for the nodes further down on the route.

In Table 2, we investigate the sensitivity of the developed algorithm to the changes of the sampling time. We consider the results from Table 1 row 7 in which σ=1, q=1, and γ=0.5. We vary T_S between 1 to 8 minutes without changing the values of σ, q, and γ, and calculate the approximate upper bound of the error variances of the estimation at each node of route *r*. The results are presented in Table 2 column 2. Route *r* is then simulated 1000 times. Table 2 columns 3 and 4 show the mean and error variance of the arrival times at the nodes of the simulated route *r*, respectively.

{*Table 2 to be added here.*}

Observe in Table 2 that, in some simulation scenarios, we have selected fairly large values for the network parameters (e.g., for T_S =1 min, the standard deviation of the travel time on each arc is $\sigma + q$ = 2 minutes within a minute of traveling along the route). Table 2 indicates that for relatively small sampling times (i.e. T_S =1, 2, and 4 min), Equation (41) results in a valid approximation of the upper bound of the error variance of the arrival times at the nodes. As seen from Table 2 the approximation does not hold when the sampling time was not small (e.g., for T_S =8 min). In other words, the developed algorithm was able to find an approximation for the upper bound of the error variance of the arrival times at the nodes for small values of T_S. As stated in Theorem 2, it results in a true upper bound of the error variance only when $T_S \to 0$.

C. *Simulation Experiment 3 (Dynamic stochastic route)*

Here, we consider a dynamic stochastic network in order to evaluate the efficiency of the developed algorithm when the network characteristics alter in time.

Considering route *r* in Figure 7, we assume that the number adjacent to arc (*i,j*) is the initial value of the travel time along that arc, i.e., $x_{ij}(0)$. Without loss of generality, we choose the value of η_{ij} in (3) such that the travel time on arc (*i,j*) has a sinusoidal-like wave form in time as shown in Figure 8.

To measure the dynamism of the network, we define and use the concept of *degree of dynamism* (DOD) for each arc. The DOD of arc (*i,j*), when $\sigma_{ij}(k)$ and $q_{ij}(k)$ are zero $\forall k = 0,1,2,\ldots$, is defined by

$$DOD_{ij} = \frac{\sup\left|x_{ij}(k) - x_{ij}(0)\right|}{x_{ij}(0)} \times 100\% . \qquad (46)$$

We assume that the DODs of all the arcs of route *r* are equal which, hereafter, is referred to as the DOD of the route. In Figure 8 the DOD of route *r* is denoted by μ.

After determining the values of $\eta_{ij}(k)$ at each time index *k*, we assume that the ratio $\sigma_{ij}(k) / \left(1 + \left|\eta_{ij}(k)\right|\right)$, and network parameters $q_{ij}(k)$, and $r_{ij}(k)$ for all arcs (*i,j*) on route *r* and $\forall k = 0,1,2,\ldots$ are constant in time, and are equal to σ, q, and γ, respectively.

{*Figure 8 to be added here.*}

To evaluate the developed estimation algorithm, we vary the uncertainties (i.e., σ, q, and γ) and the degree of dynamism (i.e., μ) of the route given in Figure 7. Table 3 columns 5 and 6 show the estimated arrival times and the approximate upper bound of the error variances of the estimation at all the nodes of route *r* for different values of μ, σ, q, and γ. Table 3 columns 7 and 8 present the mean and error variance of the simulated arrival times at the nodes of route *r* based on the results of 1000 trials. At each simulation trial, the network in Figure 7 was generated using a Guassian random number generator. We assumed that the sampling time is 5 minutes, and that the departure time from node 1 on route *r* is 0:10′.

{*Table 3 to be added here.*}

As expected and seen from Table 3, the error variance increases sharply as we estimate the arrival time at the nodes further down on the route. Observe from Table 3 that, for some simulation scenarios, our approximation of the upper bound of the error variance is not valid, e.g. μ=25%, σ=1, q=1, and γ=0.5 (i.e., Table 3 row 18), and for nodes 2 and 3.

To investigate the effect of the changes of the sampling time on the calculated approximate upper bound, we consider the scenario presented in Table 3 row 18 and vary the sampling time, T_S, between 1 to 8 minutes. We still assume that the profile of η_{ij} in time is given by Figure 8. However to adjust the values of σ, q, and γ to the changes in T_S, we multiply these parameters by a factor of $T_S/5$, where 5 (min) is the sampling time chosen originally. The results are presented in Table 4.

{*Table 4 to be added here.*}

Table 4 demonstrates that, as the sampling time becomes smaller (i.e. T_S=1 and 2 minutes), the developed algorithm provides a valid approximation of the upper bounds of the error variance. As expected, the approximation of the upper bound is not valid for higher values of the sampling time (i.e. T_S=4 and 8 minutes). Table 4 column 7 also shows that as T_S becomes smaller the mean error of the arrival time estimator (the difference between the estimated and simulated arrival times) at the nodes of route *r* goes to zero. In other words, as $T_S \to 0$, the arrival time estimator becomes an unbiased estimator.

V. Summary and Conclusions

In this paper, methodologies were developed to estimate travel times along the arcs and arrival times at the nodes of a stochastic and dynamic network in real-time. Our travel time estimator was developed based on a predictor-corrector form of the Kalman filter. It uses the real-time and historical data of travel times along an arc of a transportation network to estimate the travel times along that arc for future times. The predicted travel times are then used to estimate the arrival times at the nodes of the network. We showed that, when the sampling time goes to zero, the developed arrival time estimator is unbiased. We also found an upper bound on the error variance of the estimator.

The results from the various simulation experiments confirmed our analytical findings. These experiments demonstrated that our developed methodologies were able to estimate the travel times along the arcs and the arrival times at the nodes of both stationary and non-stationary stochastic networks with relatively small errors.

The findings of this research have significant practical relevance. One of the primary criticisms of applying dynamic methods to solving routing problems is the lack of techniques that are robust in handling the inherent uncertainties of the stochastic and dynamic networks – for instance, the traffic networks in major cities. The methodologies developed in this research are simple and robust to estimate the travel times along the arcs and arrival times at the nodes for such networks in real-time. Therefore, planners can use these methods instead of the deterministic counterparts in generating less costly yet robust routes using historical data and real-time information.

APPENDIX

Throughout this appendix, we assume that $\eta_{ij}(k)$ in (3) is a bounded function in time whose bound is $\bar{\eta}_{ij} < \infty$.

Lemma 1 *Let route r start from node 1 and pass through nodes i and j in that order. Let* z_1^r *be the departure time from node 1, and* $\hat{z}_i^r$ *be an unbiased estimate of the arrival time at node i given* z_1^r*, i.e.,* $E\{\tilde{z}_i^r\} = 0$*, where* $\tilde{z}_i^r = \hat{z}_i^r - z_i^r$ *. Let also* $h(k)$ *be a discrete-time random process with* $E\{h(k)\} = \mu(k)$*,* $\forall k = 0,1,2,\ldots$*, where* $\mu(k)$ *is a bounded function for all* $k \geq z_1^r$*, then*

$$E\left\{\sum_{k=\Theta(z_1^r)}^{\Theta(z_i^r)-1} h(k)\right\} \approx \sum_{k=\Theta(z_1^r)}^{\Theta(\hat{z}_i^r)-1} \mu(k), \quad (47)$$

where the error of the approximation goes to zero as the sampling period $T_S \to 0$.

Proof:

$$\begin{aligned} E\left\{\sum_{k=\Theta(z_1^r)}^{\Theta(z_i^r)-1} h(k)\right\} &= E\left\{\sum_{k=\Theta(z_1^r)}^{\Theta(\hat{z}_i^r)-1} h(k) + \sum_{k=\Theta(\hat{z}_i^r)}^{\Theta(z_i^r)-1} h(k)\right\} \\ &= E\left\{\sum_{k=\Theta(z_1^r)}^{\Theta(\hat{z}_i^r)-1} h(k)\right\} + E\left\{\sum_{k=\Theta(\hat{z}_i^r)}^{\Theta(z_i^r)-1} h(k)\right\}, \end{aligned} \quad (48)$$

where, without loss of generality, in (48) we assumed that $\Theta(z_i^r) \geq \Theta(\hat{z}_i^r)$. The first term in (48) can be simplified as

$$E\left\{\sum_{k=\Theta(z_1^r)}^{\Theta(\hat{z}_i^r)-1} h(k)\right\} = \sum_{k=\Theta(z_1^r)}^{\Theta(\hat{z}_i^r)-1} E\{h(k)\} = \sum_{k=\Theta(z_1^r)}^{\Theta(\hat{z}_i^r)-1} \mu(k). \quad (49)$$

To compute the second term in (48), we use the fact that for any two random variables *x* and *y* and function *g*, we have [18]

$$E\{g(x,y)\} = E\{E\{g(x,y)|x\}\}. \quad (50)$$

Using (50), the second term in (48) can be written as

$$\begin{aligned} E\left\{\sum_{k=\Theta(\hat{z}_i^r)}^{\Theta(z_i^r)-1} h(k)\right\} &= E\left\{E\left\{\left(\sum_{k=\Theta(\hat{z}_i^r)}^{\Theta(z_i^r)-1} h(k)\right) \middle| z_i^r\right\}\right\} \\ &= E\left\{\sum_{k=\Theta(\hat{z}_i^r)}^{\Theta(z_i^r)-1} \mu(k)\right\}, \end{aligned} \quad (51)$$

And since $\mu(k)$ is a bounded function for all $k \geq z_1^r$, (51) can be simplified as

$$E\left\{\sum_{k=\Theta(\hat{z}_i^r)}^{\Theta(z_i^r)-1} \mu(k)\right\} \leq E\left\{\sum_{k=\Theta(\hat{z}_i^r)}^{\Theta(z_i^r)-1} \bar{\mu}\right\}. \quad (52)$$

As $T_S \to 0$, the last term in (52) can be approximated by

$$\begin{aligned} E\left\{\sum_{k=\Theta(\hat{z}_i^r)}^{\Theta(z_i^r)-1} \bar{\mu}\right\} &\approx E\left\{\int_{\hat{z}_i^r}^{z_i^r} \bar{\mu} \cdot d(T_S \cdot \tau)\right\} \\ &= \bar{\mu}.T_S \cdot E\left\{\int_{\hat{z}_i^r}^{z_i^r} d\tau\right\} = \bar{\mu}.T_S \cdot E\{z_i^r - \hat{z}_i^r\}. \\ &= \bar{\mu}.T_S \cdot E\{\tilde{z}_i^r\} = 0 \end{aligned} \quad (53)$$

Hence, using (53), (49), and (48), (47) is obtained. ♦

Theorem 1 (unbiased estimator) *If* $\hat{z}_i^r$ *is an unbiased estimator of* z_i^r *and* $T_S \to 0$*, then the arrival time estimator* $\hat{z}_j^r$ *in (38) will be an unbiased estimator of* z_j^r *.*

Proof: Let $\tilde{z}_j^r = z_j^r - \hat{z}_j^r$ denote the estimator error, and let $T_S \to 0$. The mean estimation error in (38) is

$$\begin{aligned} E\{\tilde{z}_j^r\} &= E\{z_j^r - \hat{z}_j^r\} \\ &= E\left\{z_i^r + x_{ij} - \hat{z}_i^r - \hat{x}_{ij}\left(\Theta(\hat{z}_i^r) \middle| \Theta(z_1^r)\right)\right\} \end{aligned} \quad (54)$$

Since $\hat{z}_i^r$ is an unbiased estimator of z_i^r, i.e., $E\{\tilde{z}_i^r\} = 0$, (54) can be written as,

$$\begin{aligned} E\{\tilde{z}_j^r\} &= E\left\{x_{ij} - \hat{x}_{ij}\left(\Theta(\hat{z}_i^r) \middle| \Theta(z_1^r)\right)\right\} \\ &\approx E\left\{\tilde{x}_{ij}\left(\Theta(z_1^r) \middle| \Theta(z_1^r)\right)\right\} + E\left\{\sum_{k=\Theta(z_1^r)}^{\Theta(z_i^r)-1} u_{ij}(k)\right\}, \\ &\quad + E\left\{\sum_{k=\Theta(z_1^r)}^{\Theta(z_i^r)-1} w_{ij}(k)\right\} - \sum_{k=\Theta(z_1^r)}^{\Theta(\hat{z}_i^r)-1} \eta_{ij}(k) \end{aligned} \quad (55)$$

where in deriving (55), we use (2) and (20). As discussed, the mean-squared filtered estimate $\hat{x}(k|k)$ is unbiased; i.e.,

$$E\left\{\tilde{x}_{ij}\left(\Theta(z_1^r) \middle| \Theta(z_1^r)\right)\right\} = 0. \quad (56)$$

Using Lemma 1, (3), and (4), (55) can be simplified as,

$$\begin{aligned} E\{\tilde{z}_j^r\} &\approx \sum_{k=\Theta(z_1^r)}^{\Theta(\hat{z}_i^r)-1} E\{u_{ij}(k)\} + \sum_{k=\Theta(z_1^r)}^{\Theta(\hat{z}_i^r)-1} E\{w_{ij}(k)\} - \sum_{k=\Theta(z_1^r)}^{\Theta(\hat{z}_i^r)-1} \eta_{ij}(k) \\ &= \sum_{k=\Theta(z_1^r)}^{\Theta(\hat{z}_i^r)-1} \eta_{ij}(k) - \sum_{k=\Theta(z_1^r)}^{\Theta(\hat{z}_i^r)-1} \eta_{ij}(k) = 0 \end{aligned} \quad (57)$$

which indicates that our estimator in (38) is an unbiased estimator $T_S \to 0$. ♦

Lemma 2 *Let route r start from node 1 and pass through nodes i and j, in that order. Let* z_1^r *be the departure time from node 1, and* $\hat{z}_i^r$ *be an unbiased estimate of the arrival time at node i given* z_1^r*. Let also* $h(k)$ *be a discrete-time random process with* $E\{h(k)\} = \mu(k)$*, and*

$$E\{h(l)h(k)\} = \left(\upsilon^2(k) + \mu^2(k)\right)\delta(k-l) \tag{58}$$

where $\upsilon(k)$ and $\mu(k)$ are bounded functions bounded by $\bar{\upsilon}$ and $\bar{\mu}$, respectively, for all $k \ge z_1^r$ and δ is defined in (7). Then

$$\operatorname{var}\left(\sum_{k=\Theta(z_1^r)}^{\Theta(z_i^r)-1} h(k)\right) \approx \sum_{k=\Theta(z_1^r)}^{\Theta(\hat{z}_i^r)-1} \upsilon^2(k) \tag{59}$$

where the error of the approximation goes to zero as $T_S \to 0$.

Proof: Using (58), we have

$$\begin{aligned}\operatorname{var}\left(\sum_{k=\Theta(z_1^r)}^{\Theta(z_i^r)-1} h(k)\right) &= \operatorname{var}\left(\sum_{k=\Theta(z_1^r)}^{\Theta(\hat{z}_1^r)-1} h(k) + \sum_{k=\Theta(\hat{z}_1^r)}^{\Theta(z_1^r)-1} h(k)\right) \\ &= \operatorname{var}\left(\sum_{k=\Theta(z_1^r)}^{\Theta(\hat{z}_1^r)-1} h(k)\right) + \operatorname{var}\left(\sum_{k=\Theta(\hat{z}_1^r)}^{\Theta(z_1^r)-1} h(k)\right)\end{aligned}, \tag{60}$$

where in (60) we assumed that $\Theta(z_i^r) \ge \Theta(\hat{z}_i^r)$ (The case where $\Theta(z_i^r) < \Theta(\hat{z}_i^r)$ can be derived similarly and is not discussed here). The first term in (60) can be simplified as follows

$$\operatorname{var}\left(\sum_{k=\Theta(z_1^r)}^{\Theta(\hat{z}_i^r)-1} h(k)\right) = \sum_{k=\Theta(z_1^r)}^{\Theta(\hat{z}_i^r)-1} \operatorname{var}(h(k)) = \sum_{k=\Theta(z_1^r)}^{\Theta(\hat{z}_i^r)-1} \upsilon^2(k). \tag{61}$$

In deriving (61), we used (58). Using (50), the second term in (60) can be calculated as follows

$$\begin{aligned}\operatorname{var}\left(\sum_{k=\Theta(\hat{z}_i^r)}^{\Theta(z_i^r)-1} h(k)\right) &= E\left\{\left(\sum_{k=\Theta(\hat{z}_i^r)}^{\Theta(z_i^r)-1} h(k) - E\left\{\sum_{k=\Theta(\hat{z}_i^r)}^{\Theta(z_i^r)-1} h(k)\right\}\right)^2\right\} \\ &= E\left\{E\left\{\left(\sum_{k=\Theta(\hat{z}_i^r)}^{\Theta(z_i^r)-1} h(k) - E\left\{\sum_{k=\Theta(\hat{z}_i^r)}^{\Theta(z_i^r)-1} h(k)\right\}\right)^2 \middle| z_i^r\right\}\right\} \\ &= E\left\{\sum_{k=\Theta(\hat{z}_i^r)}^{\Theta(z_i^r)-1} \upsilon^2(k)\right\}\end{aligned} \tag{62}$$

Assuming $\upsilon(k)$ is bounded for all $k \ge z_1^r$, (62) can be computed as follows

$$E\left\{\sum_{k=\Theta(\hat{z}_i^r)}^{\Theta(z_i^r)-1} \upsilon^2(k)\right\} \le E\left\{\sum_{k=\Theta(\hat{z}_i^r)}^{\Theta(z_i^r)-1} (\bar{\upsilon})^2\right\} \approx 0 \tag{63}$$

where in obtaining (63) the same procedure used in (53) was applied. Substituting (61), (62), and (63), (59) is obtained. ♦

Lemma 3 *Given the assumptions made in Lemma 2, then*

$$\begin{aligned}E\left\{\tilde{z}_i^r \cdot \left(\sum_{k=\Theta(z_1^r)}^{\Theta(z_i^r)-1} h(k)\right)\right\} &= E\left\{\tilde{z}_i^r \cdot \left(\sum_{k=\Theta(\hat{z}_i^r)}^{\Theta(z_i^r)-1} \mu(k)\right)\right\} \\ &\le \bar{\mu} \cdot T_S \cdot \operatorname{var}(\tilde{z}_i^r)\end{aligned} \tag{64}$$

Proof:

$$\begin{aligned}E\left\{\tilde{z}_i^r \cdot \left(\sum_{k=\Theta(z_1^r)}^{\Theta(z_i^r)-1} h(k)\right)\right\} &= E\left\{\tilde{z}_i^r \cdot \left(\sum_{k=\Theta(z_1^r)}^{\Theta(\hat{z}_i^r)-1} h(k) + \sum_{k=\Theta(\hat{z}_i^r)}^{\Theta(z_i^r)-1} h(k)\right)\right\} \\ &= E\left\{\tilde{z}_i^r \cdot \left(\sum_{k=\Theta(z_1^r)}^{\Theta(\hat{z}_i^r)-1} h(k)\right)\right\} + E\left\{\tilde{z}_i^r \cdot \left(\sum_{k=\Theta(\hat{z}_i^r)}^{\Theta(z_i^r)-1} h(k)\right)\right\}\end{aligned} \tag{65}$$

where in (65), and without loss of generality, we assumed $z_i^r \ge \hat{z}_i^r$. Since $\hat{z}_i^r$ is an unbiased estimate of the arrival time z_i^r, the first term in (65) is zero, i.e.,

$$E\left\{\tilde{z}_i^r \cdot \left(\sum_{k=\Theta(z_1^r)}^{\Theta(\hat{z}_i^r)-1} h(k)\right)\right\} = E\{\tilde{z}_i^r\} E\left\{\sum_{k=\Theta(z_1^r)}^{\Theta(\hat{z}_i^r)-1} h(k)\right\} = 0 \tag{66}$$

The second term in (65) can be calculated as follows

$$\begin{aligned}E\left\{\tilde{z}_i^r \cdot \left(\sum_{k=\Theta(\hat{z}_i^r)}^{\Theta(z_i^r)-1} h(k)\right)\right\} &= E\left\{E\left\{\tilde{z}_i^r \cdot \left(\sum_{k=\Theta(\hat{z}_i^r)}^{\Theta(z_i^r)-1} h(k)\right) \middle| z_i^r\right\}\right\} \\ &= E\left\{\tilde{z}_i^r \cdot \left(\sum_{k=\Theta(\hat{z}_i^r)}^{\Theta(z_i^r)-1} \mu(k)\right)\right\}\end{aligned} \tag{67}$$

Since $\mu(k)$ is a bounded function for all $k \ge z_1^r$, we have

$$E\left\{\tilde{z}_i^r \cdot \left(\sum_{k=\Theta(\hat{z}_i^r)}^{\Theta(z_i^r)-1} \mu(k)\right)\right\} \le E\left\{\tilde{z}_i^r \cdot \left(\sum_{k=\Theta(\hat{z}_i^r)}^{\Theta(z_i^r)-1} \bar{\mu}\right)\right\} \tag{68}$$

For small sampling period T_S, (68) can be approximated by

$$\begin{aligned}&E\left\{\tilde{z}_i^r \cdot \left(\sum_{k=\Theta(\hat{z}_i^r)}^{\Theta(z_i^r)-1} \bar{\mu}\right)\right\} \approx E\left\{\tilde{z}_i^r \cdot \int_{\hat{z}_i^r}^{z_i^r} \bar{\mu} \cdot d(T_S \cdot \tau)\right\} \\ &= \bar{\mu}.T_S \cdot E\left\{\tilde{z}_i^r \cdot \int_{\hat{z}_i^r}^{z_i^r} d\tau\right\} = \bar{\mu}.T_S \cdot E\{\tilde{z}_i^r \cdot \tilde{z}_i^r\} = \bar{\mu}.T_S \cdot \operatorname{var}(\tilde{z}_i^r)\end{aligned} \tag{69}$$

Therfore, as $T_S \to 0$, (65) to (69) together result in (64). ♦

Theorem 2 (Error variance bound) *If $\hat{z}_i^r$ is an unbiased estimator of z_i^r with error variance equal to $\operatorname{var}(\tilde{z}_i^r)$ and $T_S \to 0$, then the error variance of estimator $\hat{z}_j^r$ in (38) is bounded by*

$$\begin{aligned}\operatorname{var}(\tilde{z}_j^r) \le (1 + 2\bar{\eta}_{ij}) \operatorname{var}(\tilde{z}_i^r) + \operatorname{var}\left(\tilde{x}_{ij}\left(\Theta(z_1^r) \middle| \Theta(z_1^r)\right)\right) \\ + \sum_{k=\Theta(z_1^r)}^{\Theta(\hat{z}_i^r)-1} \sigma_{ij}^2(k) + \sum_{k=\Theta(z_1^r)}^{\Theta(\hat{z}_i^r)-1} q_{ij}^2(k)\end{aligned}. \tag{70}$$

Proof: The error variance of the estimator in (38) is

$$\begin{aligned}\operatorname{var}\left(\tilde{z}_j^r\right)&=\operatorname{var}\left(z_j^r-\hat{z}_j^r\right)=\operatorname{var}\left(z_i^r+x_{ij}-\hat{z}_i^r-\hat{x}_{ij}\left(\Theta\left(\hat{z}_i^r\right)\middle|\Theta\left(z_1^r\right)\right)\right)\\&=\operatorname{var}\left(\tilde{z}_i^r\right)+\operatorname{var}\left(x_{ij}-\hat{x}_{ij}\left(\Theta\left(\hat{z}_i^r\right)\middle|\Theta\left(z_1^r\right)\right)\right)\\&\quad+2\operatorname{cov}\left(\tilde{z}_i^r,\left(x_{ij}-\hat{x}_{ij}\left(\Theta\left(\hat{z}_i^r\right)\middle|\Theta\left(z_1^r\right)\right)\right)\right)\end{aligned}\tag{71}$$

The second term in (71) can be written as

$$\begin{aligned}&\operatorname{var}\left(x_{ij}-\hat{x}_{ij}\left(\Theta\left(\hat{z}_i^r\right)\middle|\Theta\left(z_1^r\right)\right)\right)\\&\quad\approx\operatorname{var}\left(\tilde{x}_{ij}\left(\Theta\left(\hat{z}_1^r\right)\middle|\Theta\left(z_1^r\right)\right)\right)\\&\quad+\operatorname{var}\left(\sum_{k=\Theta\left(z_1^r\right)}^{\Theta\left(z_i^r\right)-1}u_{ij}(k)\right)+\operatorname{var}\left(\sum_{k=\Theta\left(z_1^r\right)}^{\Theta\left(z_i^r\right)-1}w_{ij}(k)\right)\end{aligned}\tag{72}$$

where in deriving (72) we use (2) and (20). By using Lemma 2 and equations (3) and (4), (72) can be simplified as

$$\begin{aligned}&\operatorname{var}\left(x_{ij}-\hat{x}_{ij}\left(\Theta\left(\hat{z}_i^r\right)\middle|\Theta\left(z_1^r\right)\right)\right)\\&\approx\operatorname{var}\left(\tilde{x}_{ij}\left(\Theta\left(z_1^r\right)\middle|\Theta\left(z_1^r\right)\right)\right)+\sum_{k=\Theta\left(z_1^r\right)}^{\Theta\left(\hat{z}_i^r\right)-1}\sigma_{ij}^2(k)+\sum_{k=\Theta\left(z_1^r\right)}^{\Theta\left(\hat{z}_i^r\right)-1}q_{ij}^2(k)\end{aligned}\tag{73}$$

The last term in (71) can be calculated as follows

$$\begin{aligned}&\operatorname{cov}\left(\tilde{z}_i^r.\left(x_{ij}-\hat{x}_{ij}\left(\Theta\left(\hat{z}_i^r\right)\middle|\Theta\left(z_1^r\right)\right)\right)\right)\\&\quad=E\left\{\tilde{z}_i^r.\left(x_{ij}-\hat{x}_{ij}\left(\Theta\left(\hat{z}_i^r\right)\middle|\Theta\left(z_1^r\right)\right)\right)\right\}\\&\quad-E\left\{\tilde{z}_i^r\right\}E\left\{x_{ij}-\hat{x}_{ij}\left(\Theta\left(\hat{z}_i^r\right)\middle|\Theta\left(z_1^r\right)\right)\right\}\end{aligned}\tag{74}$$

Knowing that the estimator $\hat{z}_i^r$ is an unbiased estimator of z_i^r, the second term on the right hand side of (74) is zero. Therefore,

$$\begin{aligned}&\operatorname{cov}\left(\tilde{z}_i^r.\left(x_{ij}-\hat{x}_{ij}\left(\Theta\left(\hat{z}_i^r\right)\middle|\Theta\left(z_1^r\right)\right)\right)\right)\\&\quad=E\left\{\tilde{z}_i^r.\left(x_{ij}-\hat{x}_{ij}\left(\Theta\left(\hat{z}_i^r\right)\middle|\Theta\left(z_1^r\right)\right)\right)\right\}\end{aligned}\tag{75}$$

where by using (2) and (20) we have

$$\begin{aligned}&E\left\{\tilde{z}_i^r.\left(x_{ij}-\hat{x}_{ij}\left(\Theta\left(\hat{z}_i^r\right)\middle|\Theta\left(z_1^r\right)\right)\right)\right\}\\&\approx E\left\{\tilde{z}_i^r.\tilde{x}_{ij}\left(\Theta\left(z_1^r\right)\middle|\Theta\left(z_1^r\right)\right)\right\}+E\left\{\tilde{z}_i^r.\sum_{k=\Theta\left(z_1^r\right)}^{\Theta\left(z_i^r\right)-1}u_{ij}(k)\right\}\\&+E\left\{\tilde{z}_i^r.\sum_{k=\Theta\left(z_1^r\right)}^{\Theta\left(z_i^r\right)-1}w_{ij}(k)\right\}-E\left\{\tilde{z}_i^r\right\}\cdot\sum_{k=\Theta\left(z_1^r\right)}^{\Theta\left(z_i^r\right)-1}\eta_{ij}(k)\end{aligned}\tag{76}$$

Since $\tilde{z}_i^r$ and $\tilde{x}_{ij}\left(\Theta\left(z_1^r\right)\middle|\Theta\left(z_1^r\right)\right)$ are statistically uncorrelated and both are unbiased, the first and last terms in (76) are zero. Using Lemma 3, (3), and (4), the second and third terms in (76) can be calculated as follows

$$\begin{aligned}E\left\{\tilde{z}_i^r.\sum_{k=\Theta\left(z_1^r\right)}^{\Theta\left(z_i^r\right)-1}u_{ij}(k)\right\}&=E\left\{\tilde{z}_i^r.\sum_{k=\Theta\left(\hat{z}_1^r\right)}^{\Theta\left(z_i^r\right)-1}\eta_{ij}(k)\right\}\\&\le\bar{\eta}_{ij}\cdot T_S\cdot\operatorname{var}\left(\tilde{z}_i^r\right)\end{aligned}\tag{77}$$

$$E\left\{\tilde{z}_i^r.\sum_{k=\Theta\left(z_1^r\right)}^{\Theta\left(z_i^r\right)-1}w_{ij}(k)\right\}=0\tag{78}$$

Therefore by substituting (77) and (78) in (76), and subsequently in (75), we have

$$\begin{aligned}\operatorname{cov}\left(\tilde{z}_i^r.\left(x_{ij}-\hat{x}_{ij}\left(\hat{z}_i^r\middle|z_1^r\right)\right)\right)&=E\left\{\tilde{z}_i^r.\sum_{k=\Theta\left(\hat{z}_1^r\right)}^{\Theta\left(z_i^r\right)-1}\eta_{ij}(k)\right\}\\&\le\bar{\eta}_{ij}\cdot T_S\cdot\operatorname{var}\left(\tilde{z}_i^r\right)\end{aligned}\tag{79}$$

Using (71), (73) and (79), the bound on error variance of the estimator in (38) can be obtained which is given by (39).♦

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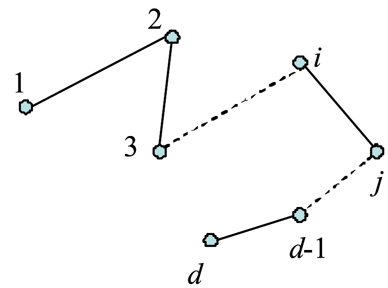

Figure 1: A typical route *r* in graph *G*.

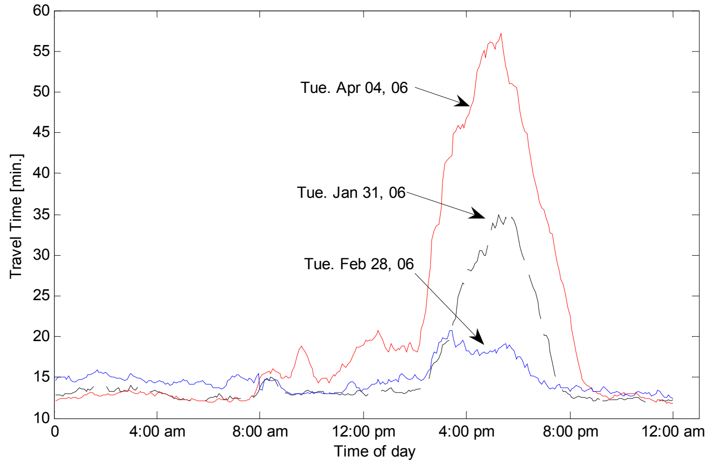

Figure 2: Travel time along Route D07 for three Tuesdays in 2006 versus time of the day.

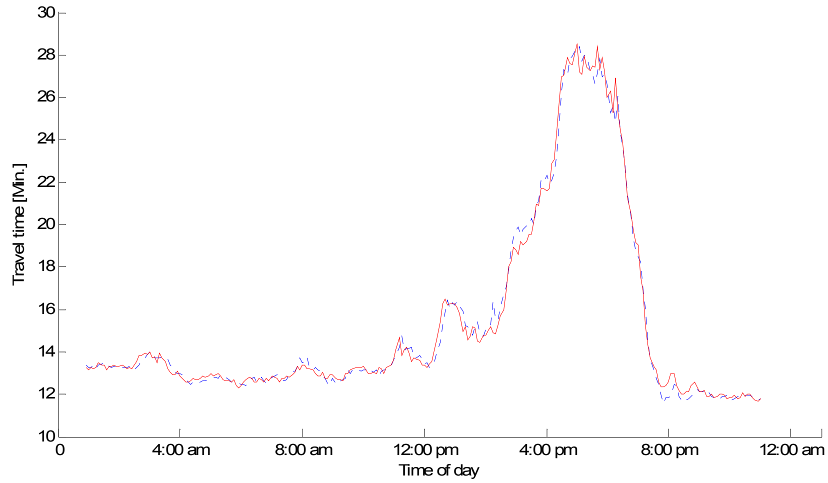

Figure 3: Actual (solid line) and single-stage predicted (dotted line) travel times for Tuesday 04/11/06.

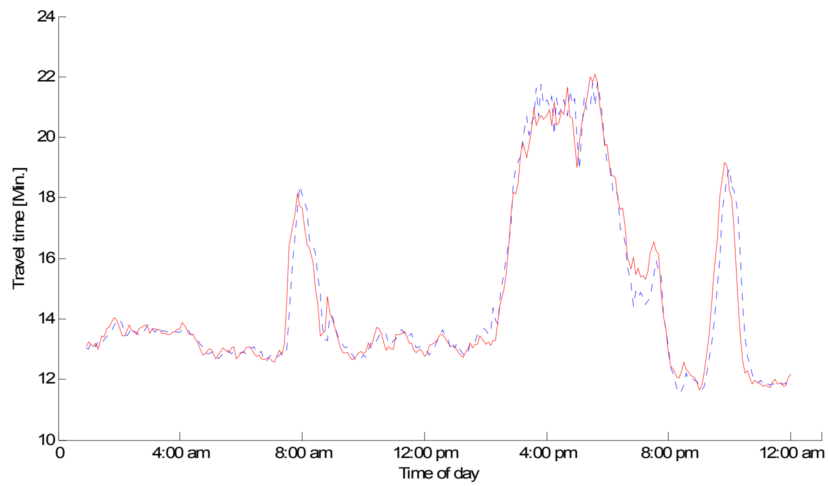

Figure 4: Actual (solid line) and single-stage predicted (dotted line) travel times for Tuesday 04/18/06.

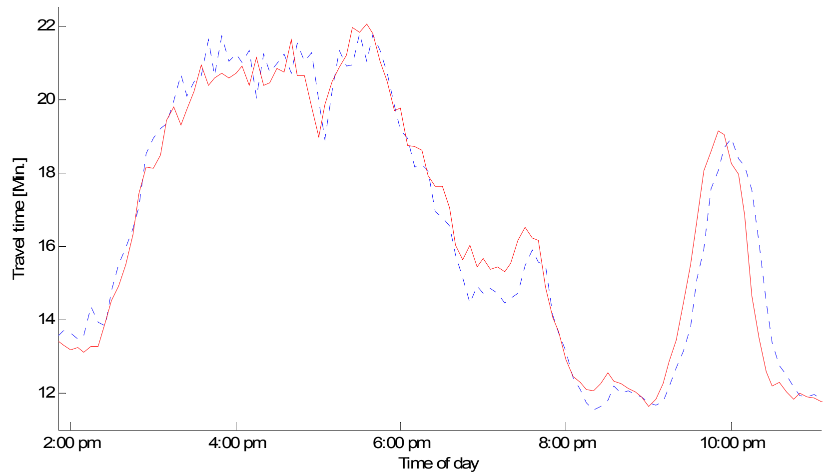

Figure 5: The zoomed-in portion of Figure 4 located between 4:00 pm and 10:00 pm. Actual (solid line) and single-stage predicted (dotted line) travel times for Tuesday 04/18/06.

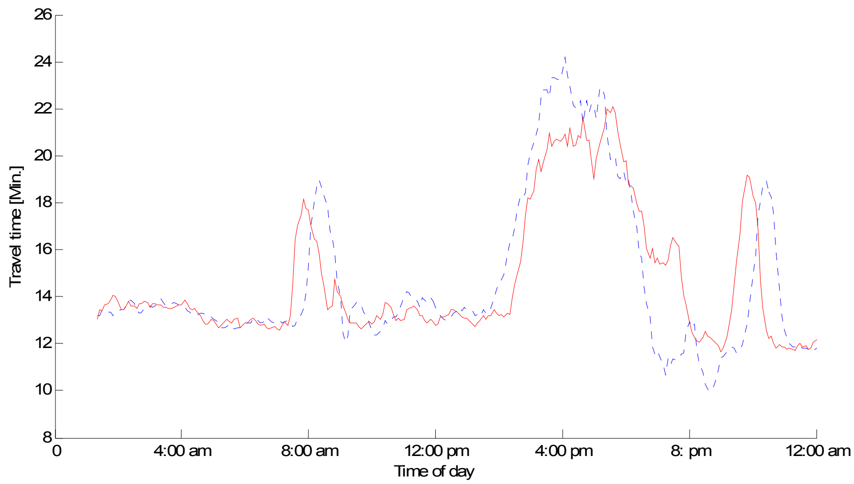

Figure 6: Actual (solid line) and 6th-stage predicted (dotted line) travel times for Tuesday 04/18/06.

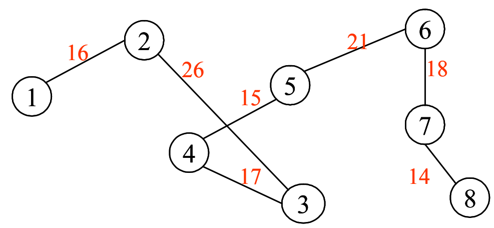

Figure 7: Simple transportation network G used in the simulation experiments.

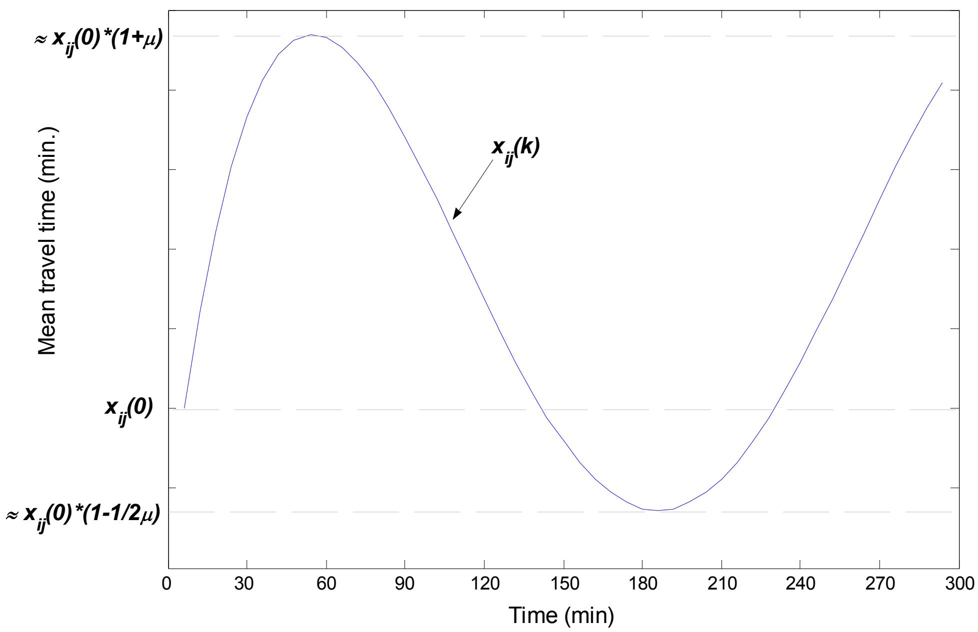

Figure 8: The assumed mean travel time on arc (i,j) when σ_{ij} and q_{ij} are zero.

Table 1: Estimated vs. simulated arrival times at the nodes of a route in a stationary stochastic network, r={1 2 3 4 5 6 7 8}.

			Estimation results		Simulation results	
σ [min.]	**q [min.]**	**γ [min.]**	**Estimated Arrival Times at Nodes [min.]**	**Approximate Upper Bound of the Error Variance of the Estimator**	**Simulated Mean Arrival Times at Nodes [min.]**	**Simulated Error Variance of Arrival Times at Nodes**
0.05	0.05	0.05	{10 26 52 69 84 105 123 137}	{0 0.1 0.5 1.4 2.6 4.1 6 8.3}	{10 26.0 52.0 69.01 84.0 105.0 123.0 137.0}	{0 0.00 0.02 0.06 0.11 0.18 0.28 0.39}
0.1	0.1	0.1	//	{0 0.2 1 2.8 5.2 8.2 12 16.6}	{10 26.0 52.0 69.02 84.03 105.0 123.0 137.1}	{0 0.01 0.09 0.29 0.54 0.84 1.23 1.68}
0.2	0.2	0.1	//	{0 0.4 2 5.6 10.4 16.4 24 33.2}	{10 26.01 52.02 68.97 83.95 104.9 123.0 137.0}	{0 0.08 0.39 1.14 1.99 3.03 4.56 6.24}
0.5	0.5	0.2	//	{0 1 5 14 26 41 60 83}	{10 26.03 52.07 68.94 83.88 104.8 122.6 136.5}	{0 0.54 2.64 7.35 13.38 20.03 30.31 41.06}
1	1	0.5	//	{0 2 10 28 52 82 120 166}	{10 25.96 51.88 68.87 83.7 104.9 123.0 136.8}	{0 1.93 9.16 25.51 50.51 77.98 110.4 154.5}
2	2	0.5	//	{ 0 4 20 56 104 16 240 332}	{10 26.0 52.1 69.09 84.1 105.4 123.8 138.1}	{0 7.7 33.5 100.2 206.7 308.1 456.7 635.2}

Table 2: Sensitivity of the arrival time estimator to changes of T_s, for σ=1, q=1, and γ=0.5 minutes

	Estimation results	**Simulation results**	
T_s [min.]	**Approximate Upper Bound of the Error Variance of the Estimator**	**Simulated Mean Arrival Times at Nodes**	**Simulated Error Variance of Arrival Times at Nodes**
1	{0 18 66 166 302 466 672 916}	{10 26.11 52.09 68.73 84.08 105.3 122.48 136.1}	{0 18.53 66.18 169.9 293.9 433.8 638.7 877.8}
2	{0 8 30 78 144 224 326 446}	{10 26.07 51.8 68.88 84.03 104.8 123.0 137.3}	{0 8.32 30.68 74.91 137.1 221.2 332.1 423.0}
4	{0 2 12 34 66 104 154 212}	{10 25.95 51.86 68.84 83.68 104.5 122.4 136.6}	{0 2.00 11.25 33.97 63.83 95.65 153.7 210.6}
8	{0 0 4 14 28 46 70 98}	{10 26.0 52.08 69.13 84.35 105.5 123.4 137.4}	{0 0 4.16 13.94 26.89 42.44 70.26 105.0}

Table 3: Estimated vs. simulated arrival times at the nodes of a route in a dynamic stochastic network, r={1 2 3 4 5 6 7 8}.

				Estimation results		Simulation results	
μ%	σ [min]	q [min]	γ [min]	Estimated Arrival Times at Nodes [min.]	Approximate Upper Bound of the Error Variance of the Estimator	Simulated Mean Arrival Times at Nodes [min.]	Simulated Error Variance of Arrival Times at Nodes
0	0.05	0.05	0.05	{10 26 52 69 84 105 123 137}	{0 0.1 0.5 1.4 2.6 4.1 6 8.3}	{10 26.0 52.0 69.0 83.98 105.0 123.0 137.0}	{0 0.00 0.02 0.06 0.12 0.19 0.28 0.40}
0	0.2	0.2	0.1	//	{0 0.4 2 5.6 10.4 16.4 24 33.2}	{10 26.0 52.0 69.01 84.02 105.0 122.9 136.9}	{0 0.07 0.42 1.13 2.18 3.51 5.00 7.28}
0	0.5	0.5	0.2	//	{0 1 5 14 26 41 60 83}	{10 26.02 52.05 69.01 84.04 105.0 123.0 136.9}	{0 0.52 2.3 7.47 13.6 21.08 29.38 41.11}
0	1	1	0.5	//	{0 2 10 28 52 82 120 166}	{10 25.9 52.0 69.32 84.29 105.2 123.0 137.0}	{0 1.9 9.3 26.2 51.73 79.15 114.8 159.5}
5	0.05	0.05	0.05	{10 26.2 53.14 70.84 86.25 107.5 125.3 139.0}	{0 0.11 0.57 1.61 3.21 5.59 8.59 11.71}	{10 26.20 53.13 70.84 86.26 107.5 125.4 139.0}	{0 0.00 0.02 0.08 0.15 0.23 0.34 0.45}
5	0.2	0.2	0.1	//	{0 0.44 2.31 6.45 12.84 22.39 34.37 46.87}	{10 26.20 53.14 70.82 86.28 107.58 125.5 139.1}	{0 0.10 0.47 1.32 2.49 3.90 5.14 7.17}
5	0.5	0.5	0.2	//	{0 1.1 5.79 16.13 32.12 55.99 85.93 117.19}	{10 26.19 53.13 70.82 86.28 107.5 125.3 138.7}	{0 0.59 3.30 8.32 15.03 22.27 32.45 44.90}
5	1	1	0.5	//	{0 2.20 11.59 32.26 64.24 111.9 171.8 234.3}	{10 26.19 53.03 70.7 85.97 107.3 125.4 139.5}	{0 2.51 12.11 30.07 55.85 91.08 134.42 186.5}
10	0.05	0.05	0.05	{10 26.40 54.28 72.69 88.51 110.0 127.5 140.9}	{0 0.12 0.66 1.85 3.83 7.28 11.72 16.05}	{10 26.40 54.28 72.7 88.5 110.0 127.7 141.0}	{0 0.00 0.03 0.08 0.16 0.28 0.34 0.44}
10	0.2	0.2	0.1	//	{0 0.48 2.65 7.43 15.33 29.12 46.91 64.21}	{10 26.39 54.3 72.74 88.59 110.0 127.7 141.0}	{0 0.12 0.61 1.44 2.5 3.88 5.58 7.54}
10	0.5	0.5	0.2	//	{0 1.20 6.63 18.58 38.33 72.81 117.2 160.5}	{10 26.43 54.15 72.37 88.09 109.5 127.1 140.4}	{0 0.76 3.68 8.37 15.58 23.03 33.86 47.46}
10	1	1	0.5	//	{0 2.40 13.27 37.16 76.66 145.6 234.5 321.0}	{10 26.38 54.3 72.82 88.59 109.9 127.7 140.7}	{0 3.18 15.89 36.27 60.28 92.98 132.0 179.3}
25	0.05	0.05	0.05	{10 27.00 57.71 78.03 94.72 116.4 132.9 145.1}	{0 0.15 0.94 2.95 6.92 16.60 30.33 42.29}	{10 27.01 57.73 78.03 94.72 116.3 132.8 145.0}	{0 0.01 0.07 0.15 0.23 0.26 0.42 0.58}
25	0.2	0.2	0.1	//	{0 0.60 3.76 11.81 27.69 66.40 121.3 169.1}	{10 27.02 57.7 77.98 94.58 116.1 132.7 144.9}	{0 0.20 1.10 2.31 3.63 5.33 7.31 9.71}
25	0.5	0.5	0.2	//	{0 1.50 9.40 29.54 69.24 166.0 303.3 422.9}	{10 26.99 57.68 77.96 94.57 116.1 132.6 144.9}	{0 1.19 7.61 13.91 22.27 33.60 44.31 58.76}
25	1	1	0.5	//	{0 3.00 18.81 59.09 138.4 332.0 606.7 845.8}	{10 26.99 57.62 77.78 94.37 116.0 132.6 144.8}	{0 4.84 28.58 56.46 87.11 130.8 185.6 239.0}
50	0.05	0.05	0.05	{10 28.0 63.43 86.51 103.3 123.6 138.0 148.0}	{0 0.20 1.48 5.67 16.77 53.55 118.9 173.7}	{10 28.0 63.42 86.48 103.3 123.7 138.0 148.0}	{0 0.02 0.14 0.27 0.45 0.67 0.87 1.06}
50	0.2	0.2	0.1	//	{0 0.80 5.93 22.68 67.09 214.2 475.6 694.8}	{10 28.01 63.5 86.46 103.4 123.7 138.0 148.0}	{0 0.41 2.34 3.90 5.49 7.51 10.45 13.28}
50	0.5	0.5	0.2	//	{0 2 14.8 56.7 167.7 535.5 1189 1737}	{10 27.95 63.44 86.35 103.5 123.8 138.0 148.4}	{0 2.68 15.95 24.93 34.94 45.49 59.94 75.47}
50	1	1	0.5	//	{0 4.0 29.7 113.4 335.5 1071 2378 3474}	{10 27.91 63.37 86.14 102.9 123.1 137.9 149.0}	{0 10.09 65.36 107.9 147.9 195.5 261.9 334.3}

Table 4: Sensitivity of the arrival time estimator to the changes of T_s; μ=25%, r={1 2 3 4 5 6 7 8}.

				Estimation results		Simulation results	
T_s	σ [min]	q [min]	γ [min]	Estimated Arrival Times at Nodes [min.]	Approximate Upper Bound of the Error Variance of the Estimator	Mean Error of Arrival Times at Nodes	Simulated Error Variance of Arrival Times at Nodes
1	0.2	0.2	0.1	{10 27.95 60.46 81.51 98.64 120.7 138.0 150.8}	{0 3.99 16.51 44.1 84.29 142.9 210.2 275.2}	{0 0.01 0.07 0.02 0.00 0.01 -0.12 -0.1}	{0 0.92 3.83 8.35 13.59 18.93 27.34 37.59}
2	0.4	0.4	0.2	{10 27.70 59.68 80.59 97.56 119.5 136.6 149.2}	{0 3.88 17.54 48.12 96.84 179.6 276.3 358.2}	{0 -0.03 0.2 0.01 -0.07 -0.1 0.15 0.23}	{0 1.96 8.65 19.1 30.03 44.45 62.61 82.92}
4	0.8	0.8	0.4	{10 26.85 57.8 78.3 95.13 116.9 133.6 145.9}	{0 2.28 16.37 51.99 117.6 262 452.6 609.6}	{0 -0.01 0.11 0.07 0.00 0.07 -0.03 0.41}	{0 2.74 18.27 39.86 63.95 84.45 121 168.6}
8	1.6	1.6	0.8	{10 26 55.56 75.48 91.81 112.9 128.9 140.6}	{0 0 12.1 49.6 143.8 438.8 978.7 1518}	{0 0 -0.02 0.06 0.82 1.1 1.49 1.8}	{0 0 46.14 97.7 149.4 230.5 332.3 434.6}