CS 561: Artificial Intelligence

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This class will use http://www.uscden.net/ and class webpage

- Up to date information
- Lecture notes
- Relevant dates, links, etc.

Course material:
[AIMA] Artificial Intelligence: A Modern Approach,
by Stuart Russell and Peter Norvig. (2nd ed)

Outline [AIMA Ch 13]

- Uncertainty
- Probability
- Syntax and Semantics
- Inference
- Independence and Bayes' Rule

Uncertainty

- Let action A_t = leave for airport t minutes before flight
- Will A_t get me there on time?

- Problems:
  - 1) partial observability (road state, other drivers' plans, etc.)
  - 2) noisy sensors (KCBS traffic reports)
  - 3) uncertainty in action outcomes (at tire, etc.)
  - 4) immense complexity of modeling and predicting traffic
- Hence a purely logical approach either
  - 1) risks falsehood: A_{25} will get me there on time"

  or 2) leads to conclusions that are too weak for decision making:

  "A_{25} will get me there on time if there's no accident on the bridge

  and it doesn't rain and my tires remain intact etc., etc."

- (A_{1440} might reasonably be said to get me there on time but I'd have to stay overnight in the airport …)

Methods for handling uncertainty

- Default or nonmonotonic logic:

  Assume my car does not have a flat tire

  Assume A_{25} works unless contradicted by evidence
- Issues: What assumptions are reasonable? How to handle contradiction?

- Rules with fudge factors:

  $A_{25} \rightarrow_{0.3} AtAirportOnTime$

  $Sprinkler \rightarrow_{0.99} WetGrass$

  $WetGrass \rightarrow_{0.7} Rain$
- Issues: Problems with combination, e.g., $Sprinkler$ causes $Rain$??

- Probability

  Given the available evidence,

  A_{25} will get me there on time with probability 0.04
- Mahaviracarya (9th C.), Cardamo (1565) theory of gambling

- (Fuzzy logic handles degree of truth NOT uncertainty e.g., $WetGrass$ is true to degree 0.2)

Probability

Probabilistic assertions summarize effects of

laziness: failure to enumerate exceptions, qualications, etc.

ignorance: lack of relevant facts, initial conditions, etc.

Subjective or Bayesian probability:

Probabilities relate propositions to one's own state of knowledge

e.g., $P(A_{25}|\text{no reported accidents}) = 0.06$

- These are not claims of a "probabilistic tendency" in the current situation (but might be learned from past experience of similar situations)

- Probabilities of propositions change with new evidence:

  e.g., $P(A_{25}|\text{no reported accidents, 5 a.m.}) = 0.15$

- (Analogous to logical entailment status $KB \models \alpha$, not truth.)

Making decisions under uncertainty

- Suppose I believe the following:

$$P(A_{25} \text{ gets me there on time} \mid \ldots) = 0.04$$
$$P(A_{90} \text{ gets me there on time} \mid \ldots) = 0.70$$
$$P(A_{120} \text{ gets me there on time} \mid \ldots) = 0.95$$
$$P(A_{1440} \text{ gets me there on time} \mid \ldots) = 0.9999$$

- Which action to choose?
- Depends on my preferences for missing flight vs. airport cuisine, etc.
- Utility theory is used to represent and infer preferences
- Decision theory = utility theory + probability theory

Probability basics

Begin with a set Ω—the sample space

e.g., 6 possible rolls of a die.

$\omega \in \Omega$ is a sample point/possible world/atomic event

A probability space or probability model is a sample space with an assignment $P(\omega)$ for every $\omega \in \Omega$ s.t.

$0 \leq P(\omega) \leq 1$

$\Sigma_\omega P(\omega) = 1$

e.g., $P(1) = P(2) = P(3) = P(4) = P(5) = P(6) = 1/6$.

An event A is any subset of Ω

$$P(A) = \Sigma_{\{\omega \in A\}} P(\omega)$$

E.g., $P(\text{die roll} < 4) = P(1) + P(2) + P(3) = 1/6 + 1/6 + 1/6 = 1/2$

Random variables

A random variable is a function from sample points to some range, e.g., the reals or Booleans

e.g., $Odd(1) = true$.

P induces a probability distribution for any r.v. X:

$$P(X = x_i) = \Sigma_{\{\omega : X(\omega) = x_i\}} P(\omega)$$

e.g., $P(Odd = true) = P(1) + P(3) + P(5) = 1/6 + 1/6 + 1/6 = 1/2$

Propositions

- Think of a proposition as the event (set of sample points) where the proposition is true
- Given Boolean random variables A and B:

  event a = set of sample points where $A(\omega)=true$

  event $\neg a$ = set of sample points where $A(\omega)=false$

  event $a \wedge b$ = points where $A(\omega)=true$ and $B(\omega)=true$
- Often in AI applications, the sample points are defined by the values of a set of random variables, i.e., the sample space is the Cartesian product of the ranges of the variables

- With Boolean variables, sample point = propositional logic model

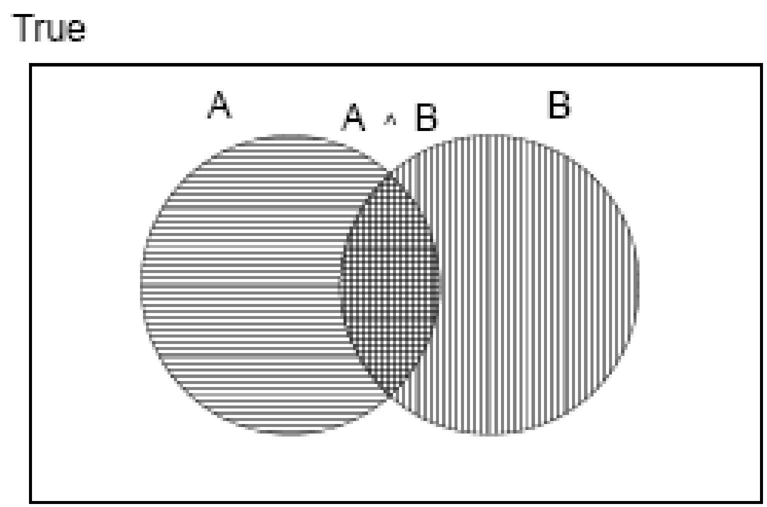
  e.g., $A=true$, $B=false$, or $a \wedge \neg b$.
- Proposition = disjunction of atomic events in which it is true

  e.g., $(a \vee b) \equiv (\neg a \wedge b) \vee (a \wedge \neg b) \vee (a \wedge b)$

  $\Rightarrow P(a \vee b) = P(\neg a \wedge b) + P(a \wedge \neg b) + P(a \wedge b)$

Why use probability?

- The definitions imply that certain logically related events must have related probabilities
- E.g., $P(a \vee b) = P(a) + P(b) + P(a \wedge b)$



- de Finetti (1931): an agent who bets according to probabilities that violate these axioms can be forced to bet so as to lose money regardless of outcome.

Syntax for propositions

- Propositional or Boolean random variables
  - e.g., $Cavity$ (do I have a cavity?)
  - $Cavity = true$ is a proposition, also written $cavity$
- Discrete random variables (finite or infinite)
  - e.g., $Weather$ is one of $\langle sunny, rain, cloudy, snow \rangle$
  - $Weather = rain$ is a proposition
  - Values must be exhaustive and mutually exclusive
- Continuous random variables (bounded or unbounded)
  - e.g., $Temp = 21.6$; also allow, e.g., $Temp < 22.0$.
- Arbitrary Boolean combinations of basic propositions

Prior probability

- Prior or unconditional probabilities of propositions
  e.g., $P(Cavity = true) = 0{:}1$ and $P(Weather = sunny) = 0{:}72$
  correspond to belief prior to arrival of any (new) evidence

- Probability distribution gives values for all possible assignments:
  $P(Weather) = \langle 0.72, 0.1, 0.08, 0.1 \rangle$ (normalized, i.e., sums to 1)

- Joint probability distribution for a set of r.v.s gives the probability of every atomic event on those r.v.s (i.e., every sample point)
  $P(Weather, Cavity)$ = a 4 2 matrix of values:

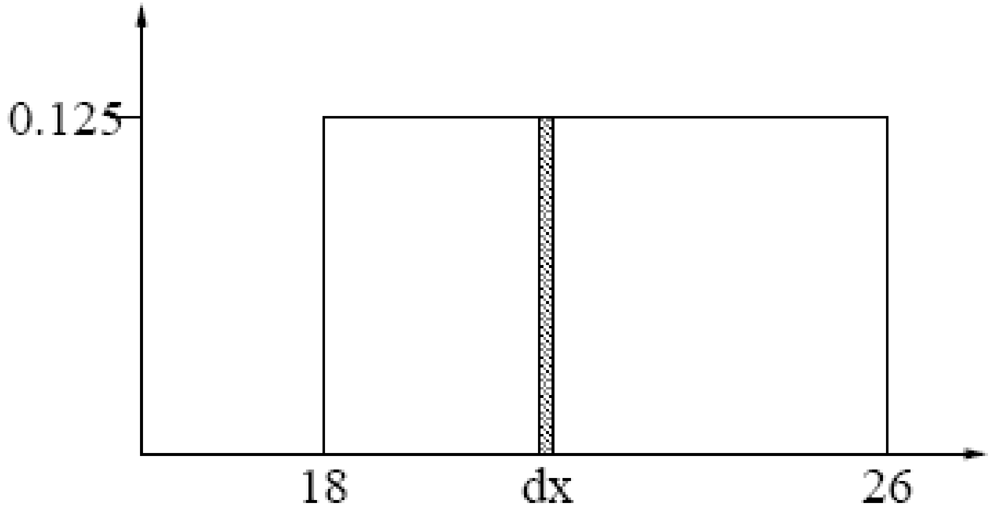
$Weather =$	$sunny$	$rain$	$cloudy$	$snow$
$Cavity = true$	0.144	0.02	0.016	0.02
$Cavity = false$	0.576	0.08	0.064	0.08

- Every question about a domain can be answered by the joint distribution because every event is a sum of sample points

Probability for continues variables

- Express distribution as a parameterized function of value:

  $P(X = x) = U[18; 26](x)$ = uniform density between 18 and 26

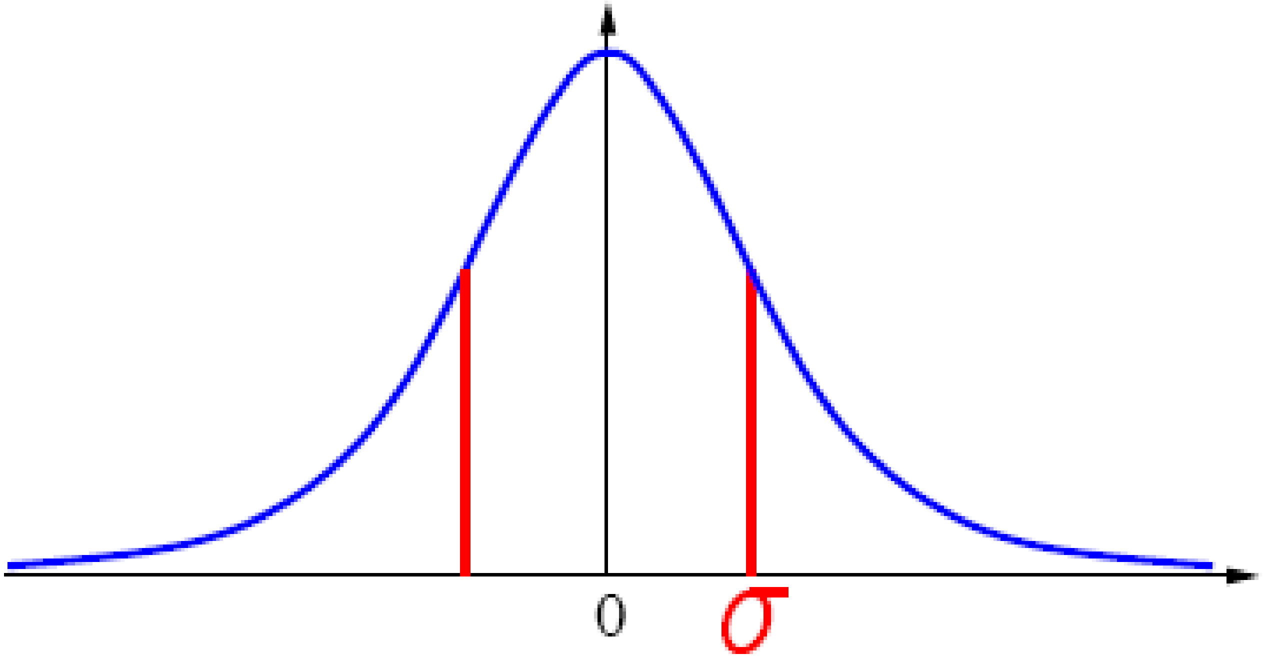


- Here P is a density; integrates to 1.
- $P(X = 20.5) = 0.125$ really means

  $lim_{dx \to 0} P(20.5 \leq X \leq 20.5 + dx)/dx = 0.125$

Gaussian density

$$P(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2}$$



Conditional probability

- Conditional or posterior probabilities
  e.g., $P(cavity|toothache) = 0.8$
  i.e., given that $toothache$ is all I know
  NOT "if $toothache$ then 80% chance of $cavity$"

- (Notation for conditional distributions:
  $\mathbf{P}(Cavity|Toothache)$ = 2-element vector of 2-element vectors)

- If we know more, e.g., $cavity$ is also given, then we have
  $P(cavity|toothache, cavity) = 1$
- Note: the less specific belief remains valid after more evidence arrives, but is not always useful

- New evidence may be irrelevant, allowing simplification, e.g.,
  $P(cavity|toothache, 49ersWin) = P(cavity|toothache) = 0{:}8$
- This kind of inference, sanctioned by domain knowledge, is crucial

Conditional Probability

Definition of conditional probability:

$$P(a|b) = \frac{P(a \wedge b)}{P(b)} \text{ if } P(b) \neq 0$$

Product rule gives an alternative formulation:

$$P(a \wedge b) = P(a|b)P(b) = P(b|a)P(a)$$

A general version holds for whole distributions, e.g.,

$$\mathbf{P}(Weather, Cavity) = \mathbf{P}(Weather|Cavity)\mathbf{P}(Cavity)$$

(View as a 4×2 set of equations, not matrix mult.)

Chain rule is derived by successive application of product rule:

$$\begin{aligned}\mathbf{P}(X_1, \ldots, X_n) &= \mathbf{P}(X_1, \ldots, X_{n-1})\ \mathbf{P}(X_n|X_1, \ldots, X_{n-1}) \\ &= \mathbf{P}(X_1, \ldots, X_{n-2})\ \mathbf{P}(X_{n_1}|X_1, \ldots, X_{n-2})\ \mathbf{P}(X_n|X_1, \ldots, X_{n-1}) \\ &= \ldots \\ &= \Pi_{i=1}^{n} \mathbf{P}(X_i|X_1, \ldots, X_{i-1})\end{aligned}$$

Inference by enumeration

Start with the joint distribution:

	toothache		*¬ toothache*	
	catch	*¬ catch*	*catch*	*¬ catch*
cavity	.108	.012	.072	.008
¬ cavity	.016	.064	.144	.576

For any proposition ϕ, sum the atomic events where it is true:

$P(\phi) = \sum_{\omega:\omega\models\phi} P(\omega)$

Inference by enumeration

Start with the joint distribution:

	$toothache$		$\neg$ $toothache$	
	$catch$	$\neg$ $catch$	$catch$	$\neg$ $catch$
$cavity$	.108	.012	.072	.008
$\neg$ $cavity$	.016	.064	.144	.576

For any proposition ϕ, sum the atomic events where it is true:

$$P(\phi) = \Sigma_{\omega:\omega \models \phi} P(\omega)$$

$$P(toothache) = 0.108 + 0.012 + 0.016 + 0.064 = 0.2$$

Inference by enumeration

Start with the joint distribution:

	toothache		$\neg$ *toothache*	
	catch	$\neg$ *catch*	*catch*	$\neg$ *catch*
cavity	.108	.012	.072	.008
$\neg$ *cavity*	.016	.064	.144	.576

For any proposition ϕ, sum the atomic events where it is true:

$$P(\phi) = \Sigma_{\omega:\omega \models \phi} P(\omega)$$

$$P(cavity \vee toothache) = 0.108 + 0.012 + 0.072 + 0.008 + 0.016 + 0.064 = 0.28$$

Inference by enumeration

Start with the joint distribution:

	toothache		¬ *toothache*	
	catch	¬ *catch*	*catch*	¬ *catch*
cavity	.108	.012	.072	.008
¬ *cavity*	.016	.064	.144	.576

Can also compute conditional probabilities:

$$P(\neg cavity|toothache) = \frac{P(\neg cavity \wedge toothache)}{P(toothache)}$$

$$= \frac{0.016 + 0.064}{0.108 + 0.012 + 0.016 + 0.064} = 0.4$$

Normalization

Start with the joint distribution:

	toothache		$\neg$ *toothache*	
	catch	$\neg$ *catch*	*catch*	$\neg$ *catch*
cavity	.108	.012	.072	.008
$\neg$ *cavity*	.016	.064	.144	.576

Denominator can be viewed as a normalization constant α

$$
\begin{aligned}
\mathbf{P}(Cavity|toothache) &= \alpha\,\mathbf{P}(Cavity, toothache) \\
&= \alpha\,[\mathbf{P}(Cavity, toothache, catch) + \mathbf{P}(Cavity, toothache, \neg catch)] \\
&= \alpha\,[\langle 0.108, 0.016\rangle + \langle 0.012, 0.064\rangle] \\
&= \alpha\,\langle 0.12, 0.08\rangle = \langle 0.6, 0.4\rangle
\end{aligned}
$$

General idea: compute distribution on query variable
by fixing evidence variables and summing over hidden variables

Inference by enumeration, contd.

- Let X be all the variables. Typically, we want
  the posterior joint distribution of the query variables Y
  given specific values e for the evidence variables E

- Let the hidden variables be $H = X - Y - E$

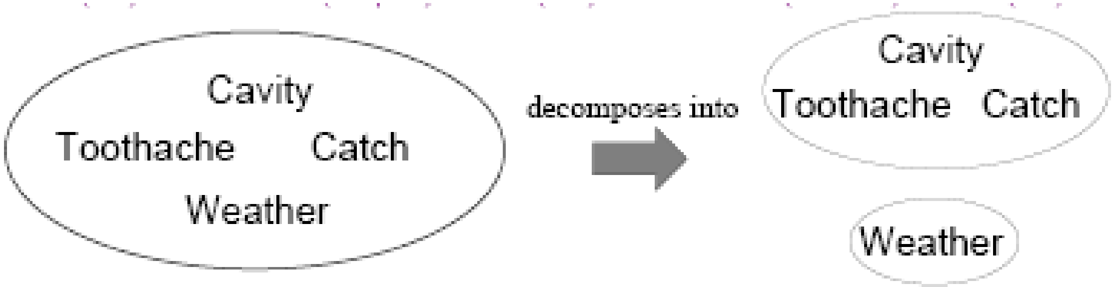
- Then the required summation of joint entries is done by summing out the hidden variables:

$$P(Y \mid E = e) = \alpha \sum_h P(Y, E = e, H = h)$$

- The terms in the summation are joint entries because Y, E, and H together exhaust the set of random variables
- Obvious problems:
  1) Worst-case time complexity $O(d^n)$ where d is the largest arity
  2) Space complexity $O(d^n)$ to store the joint distribution
  3) How to find the numbers for $O(d^n)$ entries???

Independence

- A and B are independent iff
- $P(A|B)=P(A)$ or $P(B|A)=P(B)$ or $P(A,B)= P(A)P(B)$



- $P(Toothache, Catch, Cavity, Weather) =$
  $P(Toothache, Catch, Cavity) * P(Weather)$
- 32 entries reduced to 12; for n independent biased counts, $2^n \rightarrow n$
- Absolute independence powerful but rare
- Dentristry is a large field with hundreds of variables, none of which are independent. What to do?

Conditional Independence

- $P(Toothache, Cavity, Catch)$ has $2^3 - 1 = 7$ independent entries
- If I have a cavity, the probability that the probe catches in it doesn't depend on whether I have a toothache:

  (1) $P(catch|toothache, cavity) = P(catch|cavity)$

- The same independence holds if I haven't got a cavity:

  (2) $P(catch|toothache, .cavity) = P(catch|cavity)$

- $Catch$ is conditionally independent of $Toothache$ given $Cavity$:

  $P(Catch|Toothache, Cavity) = P(Catch|Cavity)$

- Equivalent statements:

  $P(Toothache|Catch, Cavity) = P(Toothache|Cavity)$

  $P(Toothache, Catch|Cavity) = P(Toothache|Cavity)P(Catch|Cavity)$

Conditional independence contd.

- Write out full joint distribution using chain rule:
  $P(Toothache, Catch, Cavity)$
  $= P(Toothache|Catch,Cavity)P(Catch,Cavity)$
  $= P(Toothache|Catch,Cavity)P(Catch|Cavity)P(Cavity)$
  $= P(Toothache|Cavity)P(Catch|Cavity)P(Cavity)$
- I.e., $2 + 2 + 1 = 5$ independent numbers (equations 1 and 2 remove 2)

- In most cases, the use of conditional independence reduces the size of the representation of the joint distribution from exponential in n to linear in n.

- Conditional independence is our most basic and robust form of knowledge about uncertain environments.

Bayes' Rule

Product rule $P(a \wedge b) = P(a|b)P(b) = P(b|a)P(a)$

$\Rightarrow$ Bayes' rule $P(a|b) = \frac{P(b|a)P(a)}{P(b)}$

or in distribution form

$$\mathbf{P}(Y|X) = \frac{\mathbf{P}(X|Y)\mathbf{P}(Y)}{\mathbf{P}(X)} = \alpha \mathbf{P}(X|Y)\mathbf{P}(Y)$$

Useful for assessing diagnostic probability from causal probability:

$$P(Cause|Effect) = \frac{P(Effect|Cause)P(Cause)}{P(Effect)}$$

E.g., let M be meningitis, S be stiff neck:

$$P(m|s) = \frac{P(s|m)P(m)}{P(s)} = \frac{0.8 \times 0.0001}{0.1} = 0.0008$$

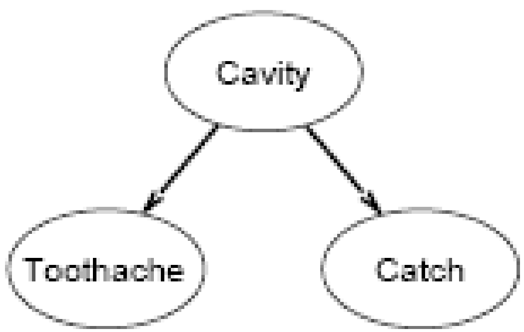
Note: posterior probability of meningitis still very small!

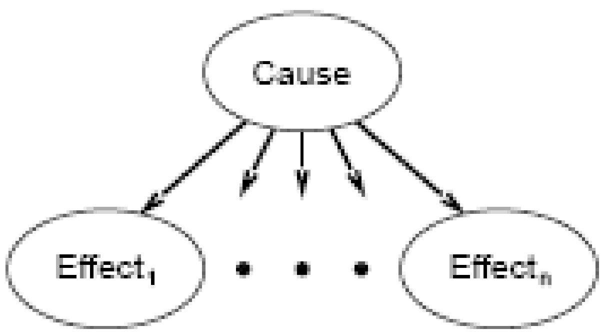
Bayes' Rule and conditional independence

$$P(Cavity|toothache \wedge catch)$$
$$= P(toothache \wedge catch|Cavity)P(Cavity)$$
$$= P(toothache|Cavity)P(catch|Cavity)P(Cavity)$$

- This is an example of a naive Bayes model:

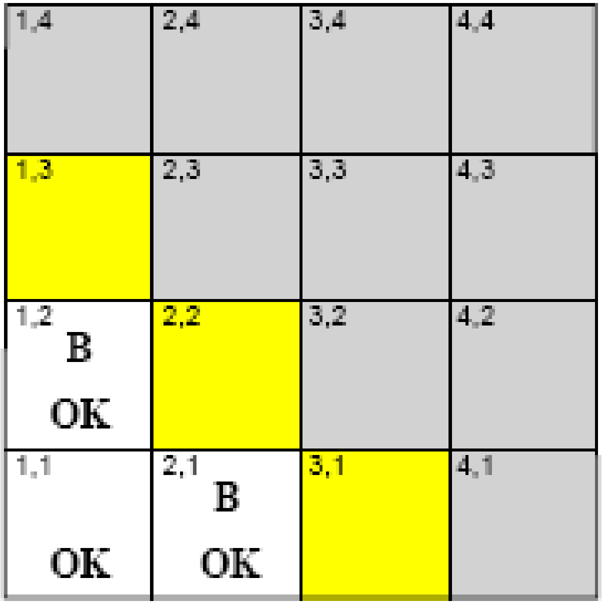
$$P(Cause, Effect_1, \ldots, Effect_n) = P(Cause)\Pi_i P(Effect_i|Cause)$$





Total number of parameters linear in n

Wumpus World



- P_{ij} $=true$ iff $[i, j]$ contains a pit
- B_{ij} $=true$ iff $[i, j]$ is breezy
- Include only $B_{1,1}, B_{1,2}, B_{2,1}$ in the probability model

Specifying the probability model

- The full joint distribution is $\mathbf{P}(P_{1,1}, \ldots, P_{4,4}, B_{1,1}, B_{1,2}, B_{2,1})$
- Apply product rule: $\mathbf{P}(B_{1,1}, B_{1,2}, B_{2,1} \mid P_{1,1}, \ldots, P_{4,4})\, \mathbf{P}(P_{1,1}, \ldots, P_{4,4})$
- (Do it this way to get $P(Effect \mid Cause)$.)
- First term: 1 if pits are adjacent to breezes, 0 otherwise
- Second term: pits are placed randomly, probability 0.2 per square:

$$\mathbf{P}(P_{1,1}, \ldots, P_{4,4}) = \Pi_{i,j=1,1}^{4,4}\, \mathbf{P}(P_{i,j}) = 0.2^{n} \times 0.8^{16-n}$$

- for n pits.

Observations and query

- We know the following facts:

$$b = \neg b_{1,1} \wedge b_{1,2} \wedge b_{2,1}$$

$$known = \neg p_{1,1} \wedge \neg p_{1,2} \wedge \neg p_{2,1}$$

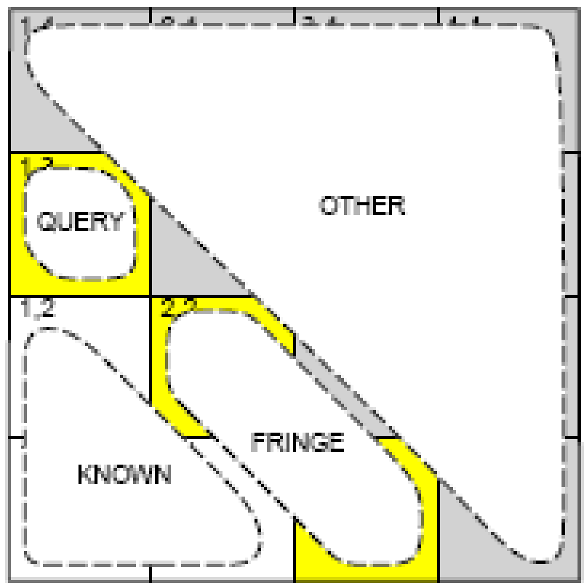
- Query is $P(P_{1,3}|known, b)$
- Define $Unknown$ = P_{ij}s other than $P_{1,3}$ and $Known$
- For inference by enumeration, we have

$$P(P_{1,3}|known, b) = \alpha \Sigma_{unknown} P(P_{1,3}, unknown, known, b)$$

- Grows exponentially with number of squares!

Using conditional independence

- Basic insight: observations are conditionally independent of other hidden squares given neighboring hidden squares



- Define $Unknown = Fringe \cup Other$
- $P(b|P_{1,3}, Known, Unknown) = P(b|P_{1,3}, Known, Fringe)$
- Manipulate query into a form where we can use this!

Using conditional independence contd.

$$\mathbf{P}(P_{1,3}|known, b) = \alpha \sum_{unknown} \mathbf{P}(P_{1,3}, unknown, known, b)$$

$$= \alpha \sum_{unknown} \mathbf{P}(b|P_{1,3}, known, unknown)\mathbf{P}(P_{1,3}, known, unknown)$$

$$= \alpha \sum_{fringe} \sum_{other} \mathbf{P}(b|known, P_{1,3}, fringe, other)\mathbf{P}(P_{1,3}, known, fringe, other)$$

$$= \alpha \sum_{fringe} \sum_{other} \mathbf{P}(b|known, P_{1,3}, fringe)\mathbf{P}(P_{1,3}, known, fringe, other)$$

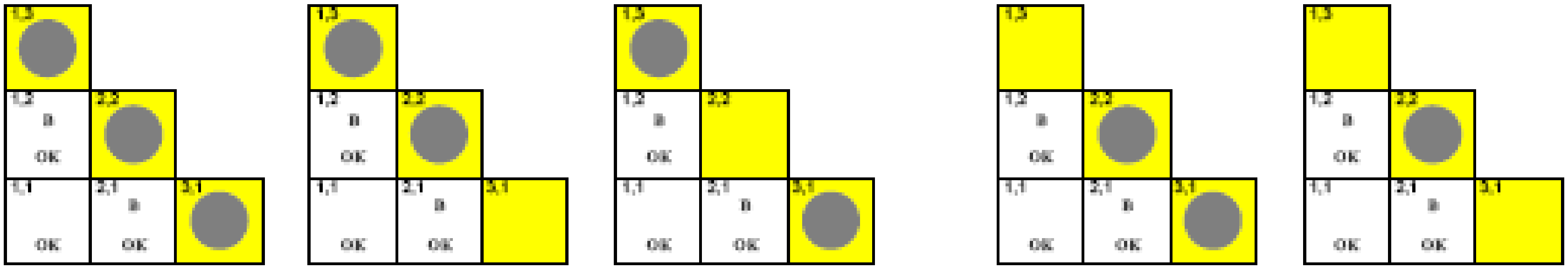
$$= \alpha \sum_{fringe} \mathbf{P}(b|known, P_{1,3}, fringe) \sum_{other} \mathbf{P}(P_{1,3}, known, fringe, other)$$

$$= \alpha \sum_{fringe} \mathbf{P}(b|known, P_{1,3}, fringe) \sum_{other} \mathbf{P}(P_{1,3})P(known)P(fringe)P(other)$$

$$= \alpha P(known)\mathbf{P}(P_{1,3}) \sum_{fringe} \mathbf{P}(b|known, P_{1,3}, fringe)P(fringe) \sum_{other} P(other)$$

$$= \alpha' \mathbf{P}(P_{1,3}) \sum_{fringe} \mathbf{P}(b|known, P_{1,3}, fringe)P(fringe)$$

Using conditional independence contd.



0.2 x 0.2 = 0.04   0.2 x 0.8 = 0.16   0.8 x 0.2 = 0.16   0.2 x 0.2 = 0.04   0.2 x 0.8 = 0.16

- $P(P_{1,3}|known, b) = \alpha' \langle 0.2(0.04+0.16+0.16), 0.8(0.04+0.16) \rangle$
  $\approx \langle 0.31, 0.69 \rangle$
- $P(P_{2,2}|known, b) \approx \langle 0.86, 0.14 \rangle$

Summary

- Probability is a rigorous formalism for uncertain knowledge
- Joint probability distribution specifies probability of every atomic event
- Queries can be answered by summing over atomic events
- For nontrivial domains, we must find a way to reduce the joint size
- Independence and conditional independence provide the tools