

Sobolev Spaces with Weights in Domains and Boundary Value Problems for Degenerate Elliptic Equations

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ABSTRACT. A family of Banach spaces is introduced to control the interior smoothness and boundary behavior of functions in a general domain. Interpolation, embedding, and other properties of the spaces are studied. As an application, a certain degenerate second-order elliptic partial differential equation is considered.

1. INTRODUCTION

Let G be a domain in \mathbb{R}^d with a non-empty boundary ∂G and $\rho_G(x) = \text{dist}(x, \partial G)$. For $1 \leq p < \infty$ and $\theta \in \mathbb{R}$ define the space $L_{p,\theta}(G)$ as follows:

$$L_{p,\theta}(G) = \left\{ u : \int_G |u(x)|^p \rho_G^{\theta-d}(x) dx < \infty \right\}.$$

Then we can define the spaces $H_{p,\theta}^m(G)$, $m = 1, 2, \dots$, so that

$$H_{p,\theta}^m(G) = \left\{ u : u, \rho_G Du, \dots, \rho_G^m D^m u \in L_{p,\theta} \right\},$$

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where D^k denotes generalized derivative of order k . The objective of the current paper is to define spaces $H_{p,\theta}^\gamma(G)$, $\gamma \in \mathbb{R}$, so that, for positive integer γ , the spaces $H_{p,\theta}^\gamma(G)$ coincide with the ones introduced above. It will be shown that these spaces can be easily defined using the spaces $H_p^\gamma(\mathbb{R}^d)$ of Bessel potentials. Note that $u \in H_{p,d-p}^1(G)$ if and only if $u/\rho_G, Du \in L_p(G)$, which means that, for bounded G , the space $H_{p,d-p}^1(G)$ coincides with the space $\overset{\circ}{H}_p^1(G)$. As a result, the spaces $H_{p,\theta}^\gamma(G)$ can be considered as a certain generalization of the usual Sobolev spaces on G with zero boundary conditions. A major application of the spaces $H_{p,\theta}^\gamma(G)$ is in the analysis of the Dirichlet problem for stochastic parabolic equations [5, 7].

Some of the spaces $H_{p,\theta}^\gamma(G)$ have been studied before. Lions and Magenes [6] introduced what corresponds to $H_{2,d}^\gamma(G)$. They constructed the scale by interpolating between the positive integer γ for $\gamma > 0$ and used duality for $\gamma < 0$. Krylov [3] defined the spaces $H_{p,\theta}^\gamma(\mathbb{R}_+^d)$, where \mathbb{R}_+^d is the half-space. After that, if G is sufficiently regular and bounded, then $H_{p,\theta}^\gamma(G)$ can be defined using the partition of unity, and this was done in [7]. Other related examples and references can be found in Chapter 3 of [10].

In this paper, an intrinsic definition (not involving \mathbb{R}_+^d) of the spaces $H_{p,\theta}^\gamma(G)$ is given for a general domain G , and the basic properties of the spaces are studied. Once a suitable definition of the spaces is found, most of the properties follow easily from the known results. Definition and properties of the spaces $H_{p,\theta}^\gamma(G)$ are presented in Sections 2, 3, and 4. Roughly speaking, the index γ controls the smoothness inside the domain, and the index θ controls the boundary behavior. In particular, the space $H_{p,\theta}^\gamma(G)$ with sufficiently large γ and $\theta < 0$ contains functions that are continuous in the closure of G and vanish on the boundary. In Section 5 some results are presented about solvability of certain degenerate elliptic equations in a general domain G .

Throughout the paper, D^m denotes a partial derivative of order m , that is, $D^m = \partial^m / \partial x_1^{m_1} \cdots \partial x_d^{m_d}$ for some $m_1 + \cdots + m_d = m$. For two Banach spaces, X, Y , notation $X \subset Y$ means that X is continuously embedded into Y .

2. DEFINITION AND MAIN PROPERTIES OF THE WEIGHTED SPACES IN DOMAINS

Let $G \subset \mathbb{R}^d$ be a domain (open connected set) with non-empty boundary ∂G , and $c > 1$, a real number. Denote by $\rho_G(x)$, $x \in G$, the distance from x to ∂G . For $n \in \mathbb{Z}$ and a fixed integer $k_0 > 0$ define the subsets G_n of G by

$$G_n = \{x \in G : c^{-n-k_0} < \rho_G(x) < c^{-n+k_0}\}.$$

Let $\{\zeta_n, n \in \mathbb{Z}\}$ be a collection of non-negative functions with the following properties:

$$\zeta_n \in C_0^\infty(G_n), \quad |D^m \zeta_n(x)| \leq N(m)c^{mn}, \quad \sum_{n \in \mathbb{Z}} \zeta_n(x) = 1.$$

The function $\zeta_n(x)$ can be constructed by mollifying the characteristic (indicator) function of G_n . If G_n is an empty set, then the corresponding ζ_n is identical zero.

If $u \in \mathcal{D}'(G)$, that is, u is a distribution on $C_0^\infty(G)$, then $\zeta_n u$ is extended by zero to \mathbb{R}^d so that $\zeta_n u \in \mathcal{D}'(\mathbb{R}^d)$. The space $H_{p,\theta}^\gamma(G)$ is defined as a collection of those $u \in \mathcal{D}'(G)$, for which $\zeta_n u$ is in H_p^γ and the norms $\|\zeta_n u\|_{H_p^\gamma}$, $n \in \mathbb{Z}$, behave in a certain way. Recall [10, Section 2.3.3] that the space of Bessel potentials H_p^γ is the closure of $C_0^\infty(\mathbb{R}^d)$ in the norm $\|\mathcal{F}^{-1}(1 + |\xi|^2)^{\gamma/2} \mathcal{F} \cdot\|_{L_p(\mathbb{R}^d)}$, where \mathcal{F} is the Fourier transform with inverse \mathcal{F}^{-1} .

Definition 2.1. Let G be a domain in \mathbb{R}^d , θ and γ , real numbers, and $p \in (1, +\infty)$. Take a collection $\{\zeta_k, n \in \mathbb{Z}\}$ as above. Then

$$H_{p,\theta}^\gamma(G) := \left\{ u \in \mathcal{D}'(G) : \|u\|_{H_{p,\theta}^\gamma(G)}^p := \sum_{n \in \mathbb{Z}} c^{n\theta} \|\zeta_{-n}(c^n \cdot) u(c^n \cdot)\|_{H_p^\gamma}^p < \infty \right\}. \quad (2.1)$$

Since $H_p^{\gamma_1} \subset H_p^{\gamma_2}$ for $\gamma_1 > \gamma_2$, the definition implies that $H_{p,\theta}^{\gamma_1}(G) \subset H_{p,\theta}^{\gamma_2}(G)$ for $\gamma_1 > \gamma_2$ and all $\theta \in \mathbb{R}$, $1 \leq p < \infty$. Still, it is necessary to establish correctness of Definition 2.1 by showing that the norms defined according to

(2.1) are equivalent for every admissible choice of the numbers c , k_0 and the functions ζ_n . Proving this equivalence is the main goal of this section.

Proposition 2.2. 1. If u is compactly supported in G , then $u \in H_{p,\theta}^\gamma(G)$ if and only if $u \in H_p^\gamma$.

2. The set $C_0^\infty(G)$ is dense in every $H_{p,\theta}^\gamma(G)$.

3. If $\gamma = m$ is a non-negative integer, then

$$H_{p,\theta}^\gamma(G) = \{u : \rho_G^k D^k u \in L_{p,\theta}(G), 0 \leq k \leq m\}, \quad (2.2)$$

where $L_{p,\theta}(G) = L_p(G, \rho_G^{\theta-d}(x)dx)$.

4. If $\{\xi_n, n \in \mathbb{Z}\}$ is a system of function so that $\xi_n \in C_0^\infty(G_n)$, $|D^m \xi_n(x)| \leq N(m)c^{mn}$, then

$$\sum_{n \in \mathbb{Z}} c^{n\theta} \|\xi_{-n}(c^n \cdot) u(c^n \cdot)\|_{H_p^\gamma}^p \leq N \|u\|_{H_{p,\theta}^\gamma}^p$$

with N independent of u , and if in addition $\sum_n \xi(x) \geq \delta > 0$ for all $x \in G$, then the reverse inequality also holds.

Proof. 1. The result is obvious because, for compactly supported u , the sum in (2.1) contains only finitely many non-zero terms.

2. Given $u \in H_{p,\theta}^\gamma(G)$, first approximate u by $u_K = u \cdot \sum_{|k| \leq K} \zeta_k$, and then mollify u_K .

3. The result follows because, for all $\nu \in \mathbb{R}$ and all x in the support of ζ_{-n} , $N_1 \leq c^{-\nu n} \rho_G^\nu(x) \leq N_2$ with N_1 and N_2 independent of n, ν, x .

4. Use that, by Theorem 4.2.2 in [9], $C_0^\infty(\mathbb{R}^d)$ functions are pointwise multipliers in every H_p^γ .

Remark 2.3. In the future we will also use a system of non-negative $C_0^\infty(\mathbb{R}^d)$ functions $\{\eta_n, n \in \mathbb{Z}\}$ with the following properties: η_n is supported in $\{x : c^{-n-k_0-1} < \rho_G(x) < c^{-n+k_0+1}\}$, $\eta(x) = 1$ on the support of ζ_n , $|D^m \eta_n(x)| \leq N(m)c^{mn}$. By Proposition 2.2(4) the functions η_n can replace ζ_n in (2.1).

Proposition 2.4. 1. For every $p \in (1, \infty)$ and $\theta, \gamma \in \mathbb{R}$, the space $H_{p,\theta}^\gamma(G)$ is a reflexive Banach space with the dual $H_{p',\theta'}^{-\gamma}(G)$, where $1/p + 1/p' = 1$ and $\theta/p + \theta'/p' = d$.

2. If $0 < \nu < 1$, $\gamma = (1 - \nu)\gamma_0 + \nu\gamma_1$, $1/p = (1 - \nu)/p_0 + \nu/p_1$, and $\theta = (1 - \nu)\theta_0 + \nu\theta_1$, then

$$H_{p,\theta}^\gamma(G) = [H_{p_0,\theta_0}^{\gamma_0}(G), H_{p_1,\theta_1}^{\gamma_1}(G)]_\nu, \quad (2.3)$$

where $[X, Y]_\nu$ is the complex interpolation space of X and Y (see [10, Section 1.9] for the definition and properties of the complex interpolation spaces).

Proof. Let $l_p^\theta(H_p^\gamma)$ be the set of sequences with elements from H_p^γ and the norm

$$\|\{f_n\}\|_{l_p^\theta(H_p^\gamma)}^p = \sum_{n \in \mathbb{Z}} c^{n\theta} \|f_n\|_{H_p^\gamma}^p.$$

Define bounded linear operators $S_{p,\theta} : H_{p,\theta}^\gamma(G) \rightarrow l_p^\theta(H_p^\gamma)$ and $R_{p,\theta} : l_p^\theta(H_p^\gamma) \rightarrow H_{p,\theta}^\gamma(G)$ as follows:

$$(S_{p,\theta}u)_n(x) = \zeta_{-n}(c^n x)u(c^n x), \quad R_{p,\theta}(\{f_n\})(x) = \sum_{n \in \mathbb{Z}} \eta_{-n}(x)f_n(c^{-n}x).$$

Note that $R_{p,\theta}S_{p,\theta} = \text{Id}_{H_{p,\theta}^\gamma(G)}$. Then, by Theorem 1.2.4 in [10], the space $H_{p,\theta}^\gamma(G)$ is isomorphic to $S_{p,\theta}(H_{p,\theta}^\gamma(G))$, which is a closed subspace of a reflexive Banach space $l_p^\theta(H_p^\gamma)$. This means that $H_{p,\theta}^\gamma(G)$ is also a reflexive Banach space. The interpolation result (2.3) follows from Theorems 1.2.4 and 1.18.1 in [10].

Denote by (\cdot, \cdot) the duality between H_p^γ and $H_{p'}^{-\gamma}$. If $v \in H_{p',\theta'}^{-\gamma}(G)$, then, by the Hölder inequality, v defines a bounded linear functional on $H_{p,\theta}^\gamma(G)$ as follows:

$$u \mapsto \langle v, u \rangle = \sum_n c^{nd}(v_n, u_n),$$

where $u_n(x) = \zeta_{-n}(c^n x)u(c^n x)$ and $v_n(x) = \eta_{-n}(c^n x)v(c^n x)$. Note that if $u, v \in C_0^\infty(G)$, then $\langle v, u \rangle = \int_G u(x)v(x)dx$.

Conversely, if V is a bounded linear functional on $H_{p,\theta}^\gamma(G)$, then we use the Hahn-Banach theorem and the equality $(l_p^\theta(H_p^\gamma))' = l_{p'}^{-\theta p'/p}(H_{p'}^{-\gamma})$ to construct $v \in H_{p',\theta'}^{-\gamma}(G)$ so that $V(u) = \langle v, u \rangle$.

One consequence of (2.3) is the interpolation inequality

$$\|u\|_{H_{p,\theta}^\gamma(G)} = \epsilon \|u\|_{H_{p,\theta_0}^{\gamma_0}(G)} + N(\nu, p, \epsilon) \|u\|_{H_{p,\theta_1}^{\gamma_1}(G)}, \quad \epsilon > 0. \quad (2.4)$$

Corollary 2.5. *The space $H_{p,\theta}^\gamma$ does not depend, up to equivalent norms, on the specific choice of the numbers c and k_0 and the functions ζ_n . Moreover, the distance function ρ_G can be replaced with any measurable function ρ satisfying $N_1\rho_G(x) \leq \rho(x) \leq N_2\rho_G(x)$ for all $x \in G$, with N_1, N_2 independent of x .*

Proof. By Proposition 2.2(3), we have the result for non-negative integer γ . For general $\gamma > 0$ the result then follows from (2.3), where we take $p_0 = p_1 = p$, $\theta_0 = \theta_1 = \theta$, and integer γ_0, γ_1 . After that, the result for $\gamma < 0$ follows by duality.

In view of Corollary 2.5, it will be assumed from now on that $c = 2$ and $k_0 = 1$.

Remark 2.6. If X is a Banach space of generalized functions on \mathbb{R}^d , then we can define the space $X_\theta(G)$ according to (2.1) by replacing the norm $\|\cdot\|_{H_p^\gamma}$ with $\|\cdot\|_X$. In particular, we can define the spaces $B_{p,q;\theta}^\gamma(G)$ and $F_{p,q;\theta}^\gamma(G)$ using the spaces $B_{p,q}^\gamma$ and $F_{p,q}^\gamma$ described in Section 2.3.1 of [10]. Results similar to Propositions 2.2 and 2.4 can then be proved in the same way.

Example. (cf. [5, Definition 1.1].) Let $G = \mathbb{R}_+^d = \{x = (x_1, \dots, x_d) \in \mathbb{R}^d : x_1 > 0\}$ and $\zeta \in C_0^\infty((b_1, b_2))$, $0 < b_1, b_2 > 3b_1$. Define $\zeta(x) = \zeta(x_1)$ and

$$H_{p,\theta}^\gamma = \left\{ u \in \mathcal{D}'(G) : \|u\|_{H_{p,\theta}^\gamma}^p := \sum_{n \in \mathbb{Z}} e^{n\theta} \|\zeta u(e^n \cdot)\|_{H_p^\gamma}^p < \infty \right\}.$$

It follows that $H_{p,\theta}^\gamma = H_{p,\theta}^\gamma(\mathbb{R}_+^d)$ with $H_{p,\theta}^\gamma(\mathbb{R}_+^d)$ defined according to (2.1), where $c = e$, $\rho_G(x) = x_1$, $\zeta_n(x) = \zeta(e^n x) / \sum_k \zeta(e^k x)$, and k_0 is the smallest positive integer for which $b_1 > e^{-k_0}$, $b_2 < e^{k_0}$.

3. POINTWISE MULTIPLIERS, CHANGE OF VARIABLES, AND LOCALIZATION

A function $a = a(x)$ is a *pointwise multiplier* in a linear normed function space X if the operation of multiplication by a is defined and continuous in X .

To describe the pointwise multipliers in the space $H_{p,\theta}^\gamma(G)$, we need some preliminary constructions. For $\gamma \in \mathbb{R}$ define $\gamma' \in [0, 1)$ as follows. If γ is an integer, then $\gamma' = 0$; if γ is not an integer, then γ' is any number from the interval $(0, 1)$ so that $|\gamma| + \gamma'$ is not an integer. The space of pointwise multipliers in H_p^γ is given by

$$B^{|\gamma|+\gamma'} = \begin{cases} L_\infty(\mathbb{R}^d), & \gamma = 0 \\ C^{n-1,1}(\mathbb{R}^d), & |\gamma| = n = 1, 2, \dots \\ C^{|\gamma|+\gamma'}(\mathbb{R}^d), & \text{otherwise,} \end{cases}$$

where $C^{n-1,1}(\mathbb{R}^d)$ is the set of functions from $C^{n-1}(\mathbb{R}^d)$ whose derivatives of order $n-1$ are uniformly Lipschitz continuous. In other words, if $u \in H_p^\gamma$ and $a \in B^{|\gamma|+\gamma'}$, then

$$\|au\|_{H_p^\gamma} \leq N(\gamma, d, p) \|a\|_{B^{|\gamma|+\gamma'}} \|u\|_{H_p^\gamma}.$$

For non-negative integer γ this follows by direct computation, for positive non-integer γ , from Corollary 4.2.2(ii) in [9], and for negative γ , by duality.

For $\nu \geq 0$, define the space $A^\nu(G)$ as follows:

- (1) if $\nu = 0$, then $A^\nu(G) = L_\infty(G)$;
- (2) if $\nu = m = 1, 2, \dots$, then

$$A^\nu(G) = \{a : a, \rho_G Da, \dots, \rho_G^{m-1} D^{m-1} a \in L_\infty(G), \rho_G^m D^m a \in C^{0,1}(G)\},$$

$$\|a\|_{A^\nu(G)} = \sum_{k=0}^{m-1} \|\rho_G^k D^k a\|_{L_\infty(G)} + \|\rho_G^m D^m a\|_{C^{0,1}(G)};$$

- (3) if $\nu = m + \delta$, where $m = 0, 1, 2, \dots$, $\delta \in (0, 1)$, then

$$A^\nu(G) = \{a : a, \rho_G Da, \dots, \rho_G^m D^m a \in L_\infty(G), \rho_G^\nu D^m a \in C^\delta(G)\},$$

$$\|a\|_{A^\nu(G)} = \sum_{k=0}^m \|\rho_G^k D^k a\|_{L_\infty(G)} + \|\rho_G^\nu D^m a\|_{C^\delta(G)}.$$

Note that, for every $a \in A^\nu(G)$ and $n \in \mathbb{Z}$,

$$\|\zeta_{-n}(2^n \cdot) a(2^n \cdot)\|_{B^\nu} \leq N \|a\|_{A^\nu(G)} \quad (3.1)$$

with N independent of n .

Theorem 3.1. *If $a \in A^{|\gamma|+\gamma'}(G)$, then*

$$\|au\|_{H_{p,\theta}^\gamma(G)} \leq N(d, \gamma, p) \|a\|_{A^{|\gamma|+\gamma'}(G)} \cdot \|u\|_{H_{p,\theta}^\gamma(G)}.$$

Proof. We have to show that $\|\eta_{-n}(2^n \cdot) a(2^n \cdot)\|_{B^{|\gamma|+\gamma'}} \leq N \|a\|_{A^{|\gamma|+\gamma'}(G)}$ with constant N independent of n . The result is obvious for $\gamma = 0$; for $|\gamma| \in (0, 1]$ it follows from the inequality (with $\delta = |\gamma| + \gamma'$)

$$\begin{aligned} |\eta_{-n}(x)a(x) - \eta_{-n}(y)a(y)| &\leq \eta_{-n}(x)\rho_G^{-\delta}(x)|a(x)\rho_G^\delta(x) - a(y)\rho_G^\delta(y)| \\ &\quad + |a(y)| |\eta_{-n}(x) - \eta_{-n}(y)| + \eta_{-n}(x)\rho_G^{-\delta}(x)|a(y)| |\rho_G^\delta(x) - \rho_G^\delta(y)| \end{aligned}$$

and the observation that both $2^n \eta_{-n}$ and ρ_G are uniformly Lipschitz continuous. If $|\gamma| > 1$, we apply the same arguments to the corresponding derivatives.

Next, we study the following question: for what mappings $\psi : G_1 \rightarrow G_2$ is the operator $u(\cdot) \mapsto u(\psi(\cdot))$ continuous from $H_{p,\theta}^\gamma(G_2)$ to $H_{p,\theta}^\gamma(G_1)$?

Theorem 3.2. *Suppose that G_1 and G_2 are domains with non-empty boundaries and $\psi : G_1 \rightarrow G_2$ is a C^1 -diffeomorphism so that $\psi(\partial G_1) = \partial G_2$. For a positive integer m define $\nu = \max(m - 1, 0)$. If $D\psi \in A^\nu(G_1)$, then, for every $\gamma \in [-\nu, m]$ and $u \in H_{p,\theta}^\gamma(G_2)$,*

$$\|u(\psi(\cdot))\|_{H_{p,\theta}^\gamma(G_1)} \leq N \|u\|_{H_{p,\theta}^\gamma(G_2)}$$

with N independent of u .

Proof. Denote by ϕ the inverse of ψ . If $\gamma = 0$, then

$$\|u(\psi(\cdot))\|_{H_{p,\theta}^\gamma(G_1)}^p = \int_{G_2} |u(y)|^p \rho_{G_1}^{\theta-d}(\phi(y)) |D\phi(y)| dy$$

and the result follows because uniform Lipschitz continuity of ρ_{G_i} , ψ , and ϕ implies that the ratio $\rho_{G_1}(\phi(x))/\rho_{G_2}(x)$ is uniformly bounded from above and below. If $\gamma = m$, the computation is similar. After that, for $\gamma \in (0, m)$, the result follows by interpolation, and for $\gamma \in [-\nu, 0)$, by duality.

The last result in this section is about localization. It answers the following question: for what collections of $C^\infty(G)$ functions $\{\xi_k, k = 1, 2, \dots\}$ are the values of $\|u\|_{H_{p,\theta}^\gamma(G)}^p$ and $\sum_n \|u\zeta_n\|_{H_{p,\theta}^\gamma(G)}^p$ comparable? To begin with, let us recall the corresponding theorem for H_p^γ .

Theorem 3.3. ([4, Lemma 6.7].) *If $\{\xi_k, k = 0, 1, \dots\}$ is a collection of $C^\infty(\mathbb{R}^d)$ functions so that $\sup_x \sum_k |D^m \xi_k(x)| \leq M(m), m \geq 0$, then $\sum_{k \geq 0} \|\xi_k v\|_{H_p^\gamma}^p \leq N \|v\|_{H_p^\gamma}^p$ with N independent of v . If in addition $\inf_x \sum_k |\xi_k(x)|^p \geq \delta$ then the reverse inequality also holds: $\|v\|_{H_p^\gamma}^p \leq N \sum_{k \geq 0} \|\xi_k v\|_{H_p^\gamma}^p$ with N independent of v .*

The following is the analogous result for $H_{p,\theta}^\gamma(G)$.

Theorem 3.4. *Suppose that $\{\chi_k, k \geq 1\}$ is a collection of $C^\infty(G)$ functions so that $\sup_{x \in G} \sum_k \rho_G^m(x) |D^m \chi_k(x)| \leq N(m), m \geq 0$. Then $\sum_k \|u \chi_k\|_{H_{p,\theta}^\gamma(G)}^p \leq N \|u\|_{H_{p,\theta}^\gamma(G)}^p$. If, in addition, $\inf_{x \in G} \sum_k |\chi_k(x)|^p \geq \delta$ for some $\delta > 0$, then $\|u\|_{H_{p,\theta}^\gamma(G)}^p \leq N \sum_k \|u \chi_k\|_{H_{p,\theta}^\gamma(G)}^p$.*

Proof. With $\hat{\chi}_{0,n} = 1 - \eta_n$, $\hat{\chi}_{k,n}(x) = \chi_k(x) \eta_{-n}(x)$, $k \geq 1$, we find

$$\sum_{k \geq 1} \|u \chi_k\|_{H_{p,\theta}^\gamma(G)}^p = \sum_{n \in \mathbb{Z}} \sum_{k \geq 0} 2^{n\theta} \|\hat{\chi}_{k,n}(2^n \cdot) \zeta_{-n}(2^n \cdot) u(2^n \cdot)\|_{H_p^\gamma}^p.$$

Both statements of the theorem now follow from Theorem 3.3.

Example. (cf. [7, Section 2].) Let G be a bounded domain of class $C^{|\gamma|+2}$ with a partition of unity $\chi_0 \in C_0^\infty(G), \chi_1, \dots, \chi_K \in C_0^\infty(\mathbb{R}^d)$ and the corresponding diffeomorphism ψ_1, \dots, ψ_K that stretch the boundary inside the support of χ_1, \dots, χ_K (see, for example, Chapter 6 of [2] for details). Then an equivalent norm in $H_{p,\theta}^\gamma(G)$ is given by

$$\|u\|_{H_{p,\theta}^\gamma(G)} = \|u \chi_0\|_{H_p^\gamma} + \sum_{m=1}^K \|u(\psi_m^{-1}(\cdot)) \chi_m(\psi_m^{-1}(\cdot))\|_{H_{p,\theta}^\gamma(\mathbb{R}_+^d)}.$$

Indeed, writing \sim to denote the equivalent norms, we deduce from Proposition 2.2(1) and Theorems 3.2 and 3.4 that

$$\|u\|_{H_{p,\theta}^\gamma(G)} \sim \sum_{m=0}^K \|u\chi_m\|_{H_{p,\theta}^\gamma(G)} \sim \|u\chi_0\|_{H_p^\gamma} + \sum_{m=1}^K \|u(\psi_m^{-1}(\cdot))\chi_m(\psi_m^{-1}(\cdot))\|_{H_{p,\theta}^\gamma(\mathbb{R}_+^d)}.$$

4. FURTHER PROPERTIES OF THE SPACES $H_{p,\theta}^\gamma(G)$

Let $\rho = \rho(x)$ be a $C^\infty(G)$ function so that $N_1\rho_G(x) \leq \rho(x) \leq N_2\rho_G(x)$ and $|\rho_G^m(x)D^{m+1}\rho(x)| \leq N(m)$ for all $x \in G$ and for every $m = 0, 1, \dots$. In particular, $\rho(x) = 0$ on ∂G and all the first-order partial derivatives of ρ are pointwise multipliers in every $H_{p,\theta}^\gamma(G)$. An example of the function ρ is

$$\rho(x) = \sum_{n \in \mathbb{Z}} 2^{-n} \zeta_n(x),$$

where the functions ζ_n are as in Section 2 with $c = 2$.

Theorem 4.1. *1. The following conditions are equivalent:*

- $u \in H_{p,\theta}^\gamma(G)$;
- $u \in H_{p,\theta}^{\gamma-1}(G)$ and $\rho Du \in H_{p,\theta}^{\gamma-1}(G)$;
- $u \in H_{p,\theta}^{\gamma-1}(G)$ and $D(\rho u) \in H_{p,\theta}^{\gamma-1}(G)$.

In addition, under either of these conditions, the norm $\|u\|_{H_{p,\theta}^\gamma(G)}$ can be replaced by $\|u\|_{H_{p,\theta}^{\gamma-1}(G)} + \|\rho Du\|_{H_{p,\theta}^{\gamma-1}(G)}$ or by $\|u\|_{H_{p,\theta}^{\gamma-1}(G)} + \|D(\rho u)\|_{H_{p,\theta}^{\gamma-1}(G)}$.

2. For every $\nu, \gamma \in \mathbb{R}$,

$$\rho^\nu H_{p,\theta}^\gamma(G) = H_{p,\theta-p\nu}^\gamma(G) \quad \text{and} \quad \|\cdot\|_{H_{p,\theta-p\nu}^\gamma(G)} \text{ is equivalent to } \|\rho^{-\nu} \cdot\|_{H_{p,\theta}^\gamma(G)}. \quad (4.1)$$

Proof. It is sufficient to repeat the arguments from the proofs of, respectively, Theorem 3.1 and Corollary 2.6 in [3].

Corollary 4.2. *1. If $u \in H_{p,\theta}^\gamma(G)$, then*

$$Du \in H_{p,\theta+p}^{\gamma-1}(G) \quad \text{and} \quad \|Du\|_{H_{p,\theta+p}^{\gamma-1}(G)} \leq N(d, \gamma, p, \theta) \|u\|_{H_{p,\theta}^\gamma(G)}.$$

2. If ρ_G is a bounded function (for example, if G is a bounded domain), then $H_{p,\theta_1}^\gamma(G) \subset H_{p,\theta_2}^\gamma(G)$ for $\theta_1 < \theta_2$ and $H_p^\gamma(G) \subset H_{p,\theta}^\gamma(G)$ for $\theta \geq d$.

Recall the following notations for continuous functions u in G :

$$\|u\|_{C(G)} = \sup_{x \in G} |u(x)|, \quad [u]_{C^\nu(G)} = \sup_{x,y \in G} \frac{|u(x) - u(y)|}{|x - y|^\nu}, \quad \nu \in (0, 1).$$

Theorem 4.3. *Assume that $\gamma - d/p = k + \nu$ for some $k = 0, 1, \dots$ and $\nu \in (0, 1)$. If $u \in H_{p,\theta}^\gamma(G)$, then*

$$\sum_{k=0}^m \|\rho^{k+\theta/p} D^k u\|_{C(G)} + [\rho^{m+\nu+\theta/p} D^m u]_{C^\nu(G)} \leq N(d, \gamma, p, \theta) \|u\|_{H_{p,\theta}^\gamma(G)}.$$

Proof. It is sufficient to repeat the arguments from the proof of Theorem 4.1 in [3].

Note that if $u \in H_{p,\theta}^\gamma(G)$ with $\gamma > 1 + d/p$ and $\theta < 0$, then, by Theorem 4.3, u is continuously differentiable in G and is equal to zero on the boundary of G . This is one reason why the spaces $H_{p,\theta}^\gamma(G)$ can be considered as a generalization of the usual Sobolev spaces with zero boundary conditions.

5. DEGENERATE ELLIPTIC EQUATIONS IN GENERAL DOMAINS

Throughout this section, $G \subset \mathbb{R}^d$ is a domain with a non-empty boundary but otherwise arbitrary, and ρ is the function introduced at the beginning of Section 4. Consider a second-order elliptic differential operator

$$\mathcal{L} = a^{ij}(x) D_i D_j + \frac{b^i(x)}{\rho(x)} D_i - \frac{c(x)}{\rho^2(x)},$$

where $D_i = \partial/\partial x_i$ and summation over the repeated indices is assumed. A related but somewhat different operator is studied in Section 6 of [10]. The objective of this section is to study solvability in $H_{p,\theta}^\gamma(G)$ of the equation $\mathcal{L}u = f$. It follows from Theorem 4.3 that, for appropriate θ and γ , the solution of the equation will also be a classical solution of the Dirichlet problem

$\mathcal{L}u = f$, $u|_{\partial G} = 0$. The values of $\gamma \in \mathbb{R}$, $1 < p < \infty$, and $\theta \in \mathbb{R}$ will be fixed throughout the section.

The following assumptions are made.

Assumption 5.1. Uniform ellipticity: there exist $\kappa_1, \kappa_2 > 0$ so that, for all $x \in G$ and $\xi \in \mathbb{R}^d$, $\kappa_1|\xi|^2 \leq a^{ij}(x)\xi_i\xi_j \leq \kappa_2|\xi|^2$.

Assumption 5.2. Regularity of the coefficients:

$$\|a\|_{A^{\nu_1}(G)} + \|b\|_{A^{\nu_2}(G)} + \|c\|_{A^{|\gamma+1|+\gamma'}(G)} \leq \kappa_2,$$

where $\nu_1 = \max(2, |\gamma-1|+\gamma')$, $\nu_2 = \max(1, |\gamma|+\gamma')$. (See beginning of Section 3 for the definition of γ' .)

Note that under assumption 5.2 the operator \mathcal{L} is bounded from $H_{p,\theta-p}^{\gamma+1}(G)$ to $H_{p,\theta+p}^{\gamma-1}(G)$. Therefore, we say that $u \in H_{p,\theta-p}^{\gamma+1}(G)$ is a solution of $\mathcal{L}u = f$ with $f \in H_{p,\theta+p}^{\gamma-1}(G)$ if the equality $\mathcal{L}u = f$ holds in $H_{p,\theta+p}^{\gamma-1}(G)$.

Theorem 5.1. *Under Assumptions 5.1 and 5.2, there exists a $c_0 > 0$ depending only on d, p, θ , the function ρ , and the coefficients a, b so that, for every $f \in H_{p,\theta+p}^{\gamma-1}(G)$ and every $c(x)$ satisfying $c(x) \geq c_0$, the equation $\mathcal{L}u = f$ has a unique solution $u \in H_{p,\theta-p}^{\gamma+1}(G)$ and $\|u\|_{H_{p,\theta-p}^{\gamma+1}(G)} \leq N\|f\|_{H_{p,\theta+p}^{\gamma-1}(G)}$ with the constant N depending only on d, γ, p, θ , the function ρ , and the coefficients a, b, c .*

To prove Theorem 5.1, we first establish the necessary a priori estimates, then prove the theorem for some special operator \mathcal{L} , and finally use the method of continuity to extend the result to more general operators.

Lemma 5.2. *If $u \in H_{p,\theta-p}^{\gamma+1}(G)$ and Assumptions 5.1 and 5.2 hold, then*

$$\|u\|_{H_{p,\theta-p}^{\gamma+1}(G)} \leq N \left(\|\mathcal{L}u\|_{H_{p,\theta+p}^{\gamma-1}(G)} + \|u\|_{H_{p,\theta-p}^{\gamma-1}(G)} \right)$$

with N independent of u .

Proof. Assume first that $b = c = 0$. Define $u_n(x) = \zeta_{-n}(2^n x)u(2^n x)$ and the operator

$$\mathcal{A}_n = (a^{ij}(2^n x)\eta_{-n}(2^n x) + (1 - \eta_{-n}(2^n x)\delta^{ij}))D_{ij},$$

where η is as in Remark 2.3. Clearly, $\|u_n\|_{H_p^{\gamma+1}} \leq N \left(\|\mathcal{A}u_n\|_{H_p^{\gamma-1}} + \|u\|_{H_p^{\gamma-1}} \right)$, and, by (3.1), N is independent of n . On the other hand,

$$\mathcal{A}_n u_n(x) = 2^{2n} (\zeta_{-n} \mathcal{L}u + 2a^{ij} D_i \zeta_{-n} D_j u + a^{ij} u D_{ij} \zeta_{-n}) (2^n x).$$

It remains to use the inequalities $\|Du\|_{H_p^{\gamma-1}} \leq N\|u\|_{H_p^\gamma} \leq \epsilon\|u\|_{H_p^{\gamma+1}} + N\epsilon^{-1}\|u\|_{H_p^{\gamma-1}}$ with sufficiently small ϵ , and then sum up the corresponding terms according to (2.1).

If b, c are not zero, then

$$\|a^{ij} D^{ij} u\|_{H_{p,\theta+p}^{\gamma-1}(G)} \leq \|\mathcal{L}u\|_{H_{p,\theta+p}^{\gamma-1}(G)} + N\|u\|_{H_{p,\theta-p}^\gamma(G)} + N\|u\|_{H_{p,\theta-p}^{\gamma-1}(G)},$$

and the result follows from the interpolation inequality (2.4).

Lemma 5.3. *If Assumptions 5.1 and 5.2 hold, then there exists a $c_0 > 0$ depending on d, p, θ , the function ρ , and the coefficients a, b , so that, for every $c(x)$ satisfying $c(x) \geq c_0$ and every $u \in L_{p,\theta}(G)$,*

$$\|u\|_{L_{p,\theta}(G)} \leq N\|\rho^2 \mathcal{L}u\|_{L_{p,\theta}(G)}$$

with N independent of u .

Proof. It is enough to consider $u \in C_0^\infty(G)$. Writing $f = -\rho^2 \mathcal{L}u$, we multiply both sides by $|u|^{p-2} u \rho^{\theta-d}$ and integrate by parts similar to the proof of Theorem 3.16 in [3]. The result is

$$\int_G f |u|^{p-2} u \rho^{\theta-d} dx = \int_G (c(x) + h(x)) |u|^p \rho^{\theta-d} dx,$$

where $|h(x)| \leq N_h$ and N_h depends on d, p, θ , and $\|a\|_{A^2(G)} + \|b\|_{A^1(G)} + \|D\rho\|_{A^1(G)}$. It remains to take $c_0 = 2N_h$ and use the Hölder inequality.

It follows from Lemmas 5.2 and 5.3 that if $c(x) \geq c_0$ and $\gamma \geq 1$, then

$$\|u\|_{H_{p,\theta-p}^{\gamma+1}(G)} \leq N\|\mathcal{L}u\|_{H_{p,\theta+p}^{\gamma-1}(G)}. \quad (5.1)$$

Lemma 5.4. *There exists a $\bar{c} > 0$ depending on p, θ, γ , and the function ρ so that the operator $\rho^2(x)\Delta - \bar{c}$ is a homeomorphism from $H_{p,\theta}^{\gamma+1}(G)$ to $H_{p,\theta}^{\gamma-1}(G)$.*

Proof. Keeping in mind that $\rho \in C^{0,1}(G)$ and $\rho(x) = 0$ on ∂G , let $\bar{\rho}$ be a $C^{0,1}(\mathbb{R}^d)$ extension of ρ so that $\bar{\rho} \in C^\infty(G - \partial G)$. Consider a family of diffusion processes $(X_t^x, x \in \mathbb{R}^d, t \geq 0)$ defined by

$$X_t^x = x + \sqrt{2} \int_0^t \bar{\rho}(X_s^x) dW_s,$$

where $(W_t, t \geq 0)$ is a standard d -dimensional Wiener process on some probability space (Ω, \mathcal{F}, P) (see, for example, Chapter V of [1] or Chapter I of [8]). Note that, by uniqueness, $X_t^x = x$ if $x \in \partial G$, and $X_t^x \in G$ for all $t > 0$ as long as $x \in G$. Theorems (3.3) and (3.9) from Chapter I of [8] imply that, with probability one, both DX_t^x and its inverse are in $C(G)$ for all $t \geq 0$. Further analysis shows that, for every $p > 1$ and every positive integer m ,

$$E\|DX_t^x\|_{A^m(G)}^p + E\|D(X_t^x)^{-1}\|_{A^m(G)}^p \leq N_1 e^{N_2 t} \quad (5.2)$$

with constants N_1 and N_2 depending on p, m .

Assume that $f \in C_0^\infty(G)$ and define

$$u(x) = -E \int_0^\infty f(X_t^x) e^{-\bar{c}t} dt.$$

By Theorem 5.8.5 in [1], there exists a $c_1 > 0$ depending only on d and $\bar{\rho}$ so that, for $\bar{c} > c_1$, the function u is twice continuously differentiable in G and $\bar{\rho}^2(x)\Delta u(x) - \bar{c}u(x) = f(x)$ for all $x \in G$. On the other hand, after repeating the proof of Theorem 3.2 and using (5.2), we conclude that there exists a c_2 depending on $d, \gamma, \bar{\rho}$ so that, for $\bar{c} > c_2$ and for every $\gamma \in \mathbb{R}$, the function u belongs to $H_{p,\theta}^\gamma(G)$ and

$$\|u\|_{H_{p,\theta}^\gamma(G)} \leq N \|f\|_{H_{p,\theta}^\gamma(G)}.$$

The statement of Lemma 5.4 now follows.

Proof of Theorem 5.1. Take \bar{c} as in Lemma 5.4 and define the operators $\mathcal{L}_0 = \Delta - \bar{c}/\rho^2(x)$ and $\bar{\mathcal{L}}_0 = \rho^2(x)\Delta - \bar{c}$. Lemmas 5.4 and Theorem 4.1(2) imply that, for all $\gamma, \theta \in \mathbb{R}$ and $1 < p < \infty$, these operators are homeomorphisms from $H_{p,\theta-p}^{\gamma+1}(G)$ to, respectively, $H_{p,\theta+p}^{\gamma-1}(G)$ and $H_{p,\theta-p}^{\gamma-1}(G)$.

Assume first that $\gamma \geq 1$. Then a priori estimate (5.1) and the method of continuity (using the operators $\lambda\mathcal{L}+(1-\lambda)\mathcal{L}_0$, $0 \leq \lambda \leq 1$) imply the conclusion of the theorem.

If $\nu < 1$, then assume first that $0 \leq \nu < 1$. For $f \in H_{p,\theta+p}^{\nu-1}(G)$, define $u = \bar{\mathcal{L}}_0\mathcal{L}^{-1}(\bar{\mathcal{L}}_0^{-1}f) - \mathcal{L}^{-1}(\bar{f})$, where $\bar{f} = (\mathcal{L}\bar{\mathcal{L}}_0 - \bar{\mathcal{L}}_0\mathcal{L})\mathcal{L}^{-1}(\bar{\mathcal{L}}_0^{-1}f)$. Direct computations show that

- $\bar{f} \in H_{p,\theta+p}^\nu(G)$ and $\|\bar{f}\|_{H_{p,\theta+p}^\nu(G)} \leq N\|f\|_{H_{p,\theta+p}^{\nu-1}(G)}$;
- u is well defined, $u \in H_{p,\theta-p}^{\nu+1}(G)$, $\|u\|_{H_{p,\theta-p}^{\nu+1}(G)} \leq N\|f\|_{H_{p,\theta+p}^{\nu-1}(G)}$, and $\mathcal{L}u = f$.

This process can be repeated as many time as necessary. Theorem 5.1 is proved.

Remark 5.5. It follows from Theorem 4.3 that, if the conditions of Theorem 5.1 hold with $\gamma > d/p + 2$ and $\theta < p$, then the function u is the classical solution of

$$a^{ij}(x)D_{ij}u + \frac{b^i(x)}{\rho(x)}D_iu - \frac{c(x)}{\rho^2(x)}u = f, \quad x \in G; \quad u|_{\partial G} = 0.$$

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