# Wiener Chaos Solution of Stochastic Evolution Equations

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# The Wiener Chaos

$(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$, $W(t) = (w_k(t),\ k \geq 1)$.
$\{m_i(s),\ i \geq 1\}$ — CONS in $L_2([0,T])$,

$\xi_{ik} = \int_0^T m_i(s)dw_k(s)$.
$J = \left\{\alpha = (\alpha_i^k,\ i,k \geq 1) \;\middle|\; |\alpha| = \sum_{i,k} \alpha_i^k < \infty\right\}$;
$\xi_\alpha = \prod_{i,k} \left( \frac{H_{\alpha_i^k}(\xi_{ik})}{\sqrt{\alpha_i^k!}} \right)$, where
$H_n(x) = (-1)^n e^{x^2/2} \frac{d^n}{dx^n} e^{-x^2/2}$. Note: if $|\alpha| = 0$, then $\xi_\alpha = 1$.

**Theorem.** *(Cameron & Martin, 1947)*

*The collection* $\{\xi_\alpha,\ \alpha\ \in J\}$ *is an orthonormal basis in* $L_2(\Omega, \mathcal{F}_T^W, \mathbb{P})$*:*

*If* $\eta \in L_2(\Omega, \mathcal{F}_T^W, \mathbb{P})$ *and* $\eta_\alpha = \mathbb{E}(\eta\xi_\alpha)$*, then*

$$\eta = \sum_{\alpha\in J} \eta_\alpha \xi_\alpha$$

*and*

$$\mathbb{E}|\eta|^2 = \sum_{\alpha\in J} \eta_\alpha^2.$$

# A Technical Lemma

**Lemma.** *Define* $\xi_\alpha(t) = \mathbb{E}(\xi_\alpha | \mathcal{F}_t^W)$*. Then*

$$d\xi_\alpha(t) = \sum_{i,k} \sqrt{\alpha_i^k}\, \xi_{\alpha^-(i,k)}(t) m_i(t) dw_k(t),$$

*where* $\alpha^-(i,k)$ *is the multi-index with the components*

$$\Big(\alpha^-(i,k)\Big)_j^l = \begin{cases} \max(\alpha_i^k - 1, 0), & \text{if } i = j \text{ and } k = l, \\ \alpha_j^l, & \text{otherwise.} \end{cases}$$

**Note:** If $|\alpha| = 0$, then $\xi_\alpha(t) = 1$ for all $t \geq 0$. Otherwise, $\xi_\alpha(0) = 0$.

# Example 1: SPDE With a Classical Solution

$$du(t,x) = 1 \cdot u_{xx}(t,x)dt + \mathbf{1} \cdot u_x(t,x)dw(t),$$
$$t > 0,\ x \in \mathbb{R}, \quad u(0,x) = \varphi(x) \in L_2(\mathbb{R}).$$
$$1 - 0.5 \cdot \mathbf{1}^2 = 0.5 > 0.$$

**Fact:** $\sup\limits_{0<t<T} \mathbb{E}\|u\|^2_{L_2(\mathbb{R})}(t)$

$$+ \frac{1}{2}\int_0^T \mathbb{E}\|u_x\|^2_{L_2(\mathbb{R})}(s)ds \le \|\varphi\|^2_{L_2(\mathbb{R})}.$$

Then $u(t,x) = \sum\limits_{\alpha \in J} u_\alpha(t,x)\xi_\alpha.$

# Example 1: The S-system

One Wiener process, so $\alpha = (\alpha_i, i \geq 1)$.

$$d\xi_\alpha(t) = \sum_i \sqrt{\alpha_i}\xi_{\alpha^-(i)}(t)m_i(t)dw(t),$$

$$\xi_\alpha(0) = I(|\alpha| = 0);$$

$$du(t,x) = u_{xx}(t,x)dt + u_x(t,x)dw(t).$$

By $\mathcal{F}_t^W$-measurability of $u$, $u_\alpha(t,x) = \mathbb{E}(u(t,x)\xi_\alpha(t))$;

$$\boxed{\dot{u}_\alpha = (u_\alpha)_{xx} + \sum_i \sqrt{\alpha_i}(u_{\alpha^-(i)})_x m_i(t),}$$

$$\boxed{u_\alpha(0,x) = \varphi(x)I(|\alpha| = 0).}$$

Also, $\sup_{0<t<T} \sum_{\alpha \in J} \|u_\alpha\|^2_{L_2(\mathbb{R})}(t)$

$$+\frac{1}{2}\sum_{\alpha \in J}\int_0^T \|(u_\alpha)_x\|^2_{L_2(\mathbb{R})}(s)ds \leq \|\varphi\|^2_{L_2(\mathbb{R})}.$$

# The WC Method

Equation $\Rightarrow$ S-System $\Rightarrow$ Solution of the S-System
$\Rightarrow$ The **Winer Chaos Solution** of the Equation as a (Formal) Fourier Series.

**Theorem.** *If the Wiener Chaos solution belongs to the traditional solution space, then it is a traditional solution.*

For $du = u_{xx}dt + u_x dw(t)$, traditional solution means $(u,\psi)(t) = (\varphi,\psi) - \int_0^t (u_x,\psi_x)(s)ds + \int_0^t (u_x,\psi)(s)dw(s)$, $\psi \in C_0^{\infty}(\mathbb{R})$.

# Example 2: SPDE Without Classical Solution

$$du(t,x) = u_{xx}(t,x)dt + 2u_x(t,x)dw(t),$$
$$t > 0,\ x \in \mathbb{R},\quad u(0,x) = \varphi(x).$$

No classical solution: $1 - 0.5 \cdot (2^2) = -1 < 0$.

**Wiener Chaos solution:** $u(t,x) = \sum_{\alpha} u_\alpha(t,x)\xi_\alpha$,

$$\dot{u}_\alpha = (u_\alpha)_{xx} + 2\sum_i \sqrt{\alpha_i}(u_{\alpha^-(i)})_x m_i(t),$$
$$u_\alpha(0,x) = \varphi(x) I(|\alpha| = 0).$$

Now $\sum_{\alpha \in J} \|u_\alpha\|^2_{L_2(\mathbb{R})}(t) = \infty$, $t > 0$, but

$$\sup_{0<t<T} \sum_{k \geq 0} 4^{-k} \sum_{|\alpha|=k} \|u_\alpha\|^2_{L_2(\mathbb{R})}(t) < \|\varphi\|^2_{L_2(\mathbb{R})}.$$

# The S-System

Hilbert spaces $\mathbf{H}_s, \mathbf{H}_f, \mathbf{H}_i$.
Operators $\mathcal{A},\ \mathcal{M}_k\ :\ \mathbf{H}_s \to \mathbf{H}_f$.
Assume

$$\dot{v}(t) = \mathcal{A}v(t) + f(t)$$

has a unique solution $v \in L_2((0,T);\mathbf{H}_s)$ for every $f \in L_2((0,T);\mathbf{H}_f)$ and $v(0) \in \mathbf{H}_i$.

$$\begin{cases} \dot{u}_\alpha(t) = \mathcal{A}u_\alpha(t) + \sum\limits_{i,k} \sqrt{\alpha_i^k}\mathcal{M}_k u_{\alpha^-(i,k)}(t) m_i(t), \\ \qquad\qquad u_\alpha(0) = u_0 I(|\alpha| = 0) \in \mathbf{H}_i. \end{cases}$$

$u(t) = \sum\limits_\alpha u_\alpha(t)\xi_\alpha$ — *Wiener Chaos solution* of
$du(t) = \mathcal{A}u(t)dt + \sum\limits_{k\geq 1} \mathcal{M}_k u(t) dw_k(t)$,
$u(0) = u_0$.

# Solving the S-System

**Theorem.** *Assume that the operator $\mathcal{A}$ generates a semi-group $(\mathbb{T}_t,\ t \geq 0)$ so that, for $t > 0$, the operators $\mathbb{T}_t$ and $\mathcal{M}_k\mathbb{T}_t$ are bounded in a Hilbert space $\mathbf{H}$. Let $u_0 \in \mathbf{H}$.*

*Then, for every $N \geq 0$ and $0 < t < T$,*

$$\sum_{|\alpha|=N} \|u_\alpha\|^2_{\mathbf{H}}(t)$$
$$= \sum_{k_1,\ldots,k_N} \int_0^t \int_0^{s_N} \ldots \int_0^{s_2}$$
$$\|\mathbb{T}_{t-s_N}\mathcal{M}_{k_N} \ldots \mathbb{T}_{s_2-s_1}\mathcal{M}_{k_1}\mathbb{T}_{s_1}u_0\|^2_{\mathbf{H}} ds_1 \ldots ds_N.$$

# Example 3

$$du(t,x) = u_{xx}(t,x)dt + b(x)u_x(t,x)dw(t),$$
$$t > 0,\ x \in \mathbb{R}, \quad u(0,x) = \varphi(x) \in L_2(\mathbb{R}),$$

$b = b(x)$— measurable, $\sup\limits_{x\in\mathbb{R}} |b(x)|^2 \le 2$.

Then $\mathcal{M} = b(x)\partial/\partial x$ and

$$\sum_{|\alpha|\le N} \|u_\alpha\|^2_{L_2(\mathbb{R})}(t)$$
$$\le \|\varphi\|^2_{L_2(\mathbb{R})} - \int_0^t \int_0^s \int_0^{s_N} \dots \int_0^{s_2}$$
$$\|\mathcal{M}\mathbb{T}_{t-s_N}\mathcal{M}\dots\mathbb{T}_{s_2-s_1}\mathcal{M}\mathbb{T}_{s_1}\varphi\|^2_{L_2(\mathbb{R})} ds_1 \dots ds_N ds.$$

In particular, $\sum\limits_\alpha \|u_\alpha\|^2_{L_2(\mathbb{R})}(t) \le \|\varphi\|^2_{L_2(\mathbb{R})}$ and $u(t) \in L_2(\Omega; L_2(\mathbb{R}))$ for all $t \in [0,T]$.

# Weighted Wiener Chaos

$\boxed{\textbf{Idea}\text{: replace } w_k \text{ with } q_k w_k.}$

$Q = \{q_1, q_2, \ldots\}$, $q_k > 0$; $Q \leq \tilde{Q}$ means $q_k \leq \tilde{q}_k$ for all $k$;

$$q^{\alpha} = \prod_{i,k} q_k^{\alpha_i^k}.$$

**Definition.** *For a Banach space $X$, the $Q$-weighted Wiener Chaos space $L_{2,Q}(\mathcal{F}_T^W; X)$ is*

$$L_{2,Q}(\mathcal{F}_T^W; X) = \left\{ (u_\alpha) : \sum_{\alpha \in J} q^{2\alpha} \|u_\alpha\|_X^2 < \infty \right\}.$$

$Q = 1 \Rightarrow L_{2,Q}(\mathcal{F}_T^W; X) = L_2(\mathcal{F}_T^W; X)$;
$Q \leq \tilde{Q} \Rightarrow L_{2,\tilde{Q}}(\mathcal{F}_T^W; X) \subseteq L_{2,Q}(\mathcal{F}_T^W; X)$.

# Example 4: $w_k \to q_k w_k$

1. (Obvious) If $u(t) = 1 + \sum_{k \geq 1} \int_0^t u(s)\,dw_k(s)$,

then $u \in L_{2,Q}(\mathcal{F}_T^W; \mathbb{R})$ for every $Q = (q_1, q_2, \ldots)$ so that
$$\sum_{k \geq 1} q_k^2 < \infty.$$

2. (Nualart-Rozovskii, 1997) If
$$du(t,x) = \Delta u(t,x)dt + u(t,x)dw(t,x),$$
$$t > 0,\ x \in \mathbb{R}^d, \quad d \geq 2,$$
then $u \in L_{2,Q}(\mathcal{F}_T^W; L_2(\mathbb{R}^d))$ for some $Q$.

**Theorem.** *Consider*

$$du(t,x) = u_{xx}(t,x)dt + b(x)u_x(t,x)dw(t),$$
$$t > 0,\ x \in \mathbb{R},\quad u(0,x) = \varphi(x) \in L_2(\mathbb{R}),$$

$b = b(x)$— *bounded, measurable;*

$Q = (q, 1, 1, \ldots)$, $q > 0$. *Then*

$$u(t) \in L_{2,Q}(\mathcal{F}_T^W; L_2(\mathbb{R})) \Leftrightarrow q^2 \sup_{x\in\mathbb{R}} |b(x)|^2 < 2.$$

*If* $\sup_{x\in\mathbb{R}} |b(x)|^2 < 2$, *then* $u(t) \in L_{2,Q}(\mathcal{F}_T^W; L_2(\mathbb{R}))$ for some $q > 1$.

*If* $\sup_{x\in\mathbb{R}} |b(x)|^2 \geq 2$, *then* $u(t) \in L_{2,Q}(\mathcal{F}_T^W; L_2(\mathbb{R}))$ for some $q < 1$.

# S-Transform

$h(t) = (h_1(t), \dots h_n(t))$;

$$\mathcal{E}(h) = \exp\left\{\sum_k \int_0^T h_k(t)dw_k(t) - \frac{1}{2}\sum_k \|h_k\|^2_{L_2((0,T))}\right\}$$

If $h_k(t) = \sum_{i=1}^{n_k} h_{ik}m_i(t)$, then $(\mathcal{E}(h))_\alpha = \prod_{i,k} \frac{h_{ik}^{\alpha_i^k}}{\sqrt{\alpha_i^k}}$; $\|\mathcal{E}(h)\|^2_{L_{2,Q}(\mathcal{F}_T^W;\mathbb{R})} = \exp\left\{\sum_{i,k} h^2_{i,k}q_k^2\right\} < \infty$ for *every* $Q$. For $u \in L_{2,Q}(\mathcal{F}_T^W; X)$ can then define $\boxed{u_h = \sum_\alpha u_\alpha(\mathcal{E}(h))_\alpha \text{ — } \textbf{S-transform} \text{ of } u}$ .

**Definition. Soft solution** *of*

$$du(t) = \mathcal{A}u(t)dt + \sum_{k\geq 1} \mathcal{M}_k u(t) dw_k(t)$$

*means* $\dot{u}_h(t) = \mathcal{A}u_h(t) + \sum_{k\geq 1} h_k(t)\mathcal{M}_k u_h(t)$*.*

**Theorem.** *If* $u \in L_{2,Q}(\mathcal{F}_T^W; X)$ *for some* $Q$*, then the soft solution is equivalent to the Wiener Chaos solution.*

- To study the properties of the solution, it is easier to work with $(u_\alpha)$ than with $u_h$.
- Can have Wiener Chaos solution that belongs to no $L_{2,Q}$.

# Example 5

$$\boxed{du(t,x) = u_x(t,x)dw(t),}$$

$$t > 0,\ \ x \in \mathbb{R}, \quad u(0,x) = \varphi(x) \in \mathcal{S}(\mathbb{R}).$$

Then

$$\sum_{|\alpha|=N} \|u_\alpha\|^2_{L_2(\mathbb{R})}(t) = \frac{t^N}{N!}\|\varphi_x^{(N)}\|^2_{L_2(\mathbb{R})}.$$

- $\|\varphi_x^{(N)}\|^2_{L_2(\mathbb{R})} \le C^N\sqrt{N!} \Rightarrow u(t) \in L_2(\Omega; L_2(\mathbb{R}))$.

- $\|\varphi_x^{(N)}\|^2_{L_2(\mathbb{R})} \sim C^N N! \Rightarrow u(t) \in L_{2,Q(t)}(\Omega; L_2(\mathbb{R}))$.

- $\|\varphi_x^{(N)}\|^2_{L_2(\mathbb{R})} \ge e^{aN^2} \Rightarrow u(t) \notin L_{2,Q}(\Omega; L_2(\mathbb{R}))$.

# References

**Wiener Chaos:** Wiener (1930's), Cameron & Martin (1947), Hida et al. (1993), Øksendal et al. (1996)

**S-transform:** Kondratiev et al. (1996)

**SODE:** Krylov & Veretennikov (1976)

**Filtering:** Wong (1981), Ocone (1983), Mikulevicius & Rozovskii (1993), Budhiraja & Kallianpur (1996)

**Flows:** LeJan and Raimond (2002)

**Soft Solutions:** Mikulevicius & Rozovskii (1994), Nualart & Rozovskii (1997), Potthoff et al. (1998)

**Navier-Stokes equation:** Mikulevicius & Rozovskii (2001)

# So What?

- Have a pretty general procedure for constructing solutions for linear equation.

- S-system is often more convenient than the underlying equation.

- Big plans for the NS equation.