Turbulent transport via SPDEs and Wiener chaos

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The Transport Equation

$$\frac{\partial \theta(t,x)}{\partial t} = -\mathbf{v}(t,x)\cdot\nabla\theta(t,x),\ t>0,\ x\in\mathbb{R}^d;$$

$\theta(0,x)=\theta_0(x)$; $\mathbf{v}=(v^1,\ldots,v^d)\in\mathbb{R}^d$, $d\geq 2$.

Laminar transport:

each v^i is Lipschitz continuous in x.

$$\theta(t,x)=\theta_0(X^x_{t,0});$$

X is the flow of $\mathbf{v}$:

$$\frac{dX^x_{s,t}}{dt}=\mathbf{v}(t,X^x_{s,t}),\ t>s,\ X^x_{s,s}=x.$$

(Characteristic equation.)

Turbulent Transport

$\mathbf{v}$ is not Lipschitz continuous in x

- Example — Kolmogorov's theory: $\mathbf{v}$ is Hölder $\approx 1/3$ for $d = 3$.

- Difficulty — Existence but no uniqueness for the flow equation (intrinsic stochasticity).

- How to find θ?

Regularization

- Introducing viscosity (κ-limit):

$$\frac{\partial \theta^{\kappa}(t,x)}{\partial t} = \kappa \Delta \theta^{\kappa}(t,x) - \mathbf{v}(t,x) \cdot \nabla \theta^{\kappa}(t,x),\ t > 0,\ x \in \mathbb{R}^d;$$

$$dX_{s,t}^{\kappa,x} = \mathbf{v}(t, X_{s,t}^{\kappa,x})dt + \sqrt{2\kappa}dw(t),\ t > s,\ X_{s,s}^{\kappa,x} = x.$$

(Gawędzki and Vergassola (2000), Weinan E and Eric Vanden Eijnden (2000))

Unique strong solution of the flow equation!

- Smoothing out $\mathbf{v}$ (ε-limit):

$$\mathbf{v}^{\varepsilon}(t,x) = \frac{1}{\varepsilon^d} \int_{\mathbb{R}^d} \mathbf{v}(t,y) \psi\left(\frac{x-y}{\varepsilon}\right) dy$$

(Weinan E and Eric Vanden Eijnden (2000))

Generalized Kraichnan's Model

Physical Model for $\mathbf{v}$:

- $\mathbf{v}$ is a statistically homogeneous, isotropic, and stationary Gaussian vector field with zero mean and covariance

$$E(v^i(t,x)v^j(s,y)) = \delta(t-s)C^{ij}(x-y).$$

- For small x, $C^{ij}(x) \sim (\delta_{ij} - c^{ij}|x|^\gamma)$, $0 < \gamma < 2$.
- Div-free and rotation-free components of $\mathbf{v}$.
- Classification of flows in κ- and ε-limits. Limits can be different!

Generalized Kraichnan's Model

Mathematical Model for $\mathbf{v}$.

(LeJan and Raimond (2002))

- The matrix C is characterized by its Fourier transform:

$$\widehat{\mathbf{C}}(z) = \frac{A_0}{(1+|z|^2)^{(d+\gamma)/2}} \left(a \frac{zz^*}{|z|^2} + \frac{b}{d-1} \left(I - \frac{zz^*}{|z|^2} \right) \right),$$

- $a = 0 \Rightarrow \nabla \cdot \mathbf{v} = 0$
  (''b''— div-free component);
- $b = 0 \Rightarrow \mathbf{v} = \nabla V$ for some scalar V
  (''a''—rotation-free component).
- H_C — Reproducing kernel Hilbert space for C.
  $H_C = H^{(d+\gamma)/2}(\mathbb{R}^d; \mathbb{R}^d)$ if $a, b > 0$.

- Representation of $\mathbf{v}$:

$$v^i(t,x) = \sum_{k\geq 1} \sigma_k^i(x)\dot{w}_k(t);$$

$\dot{w}_k(t)$ are independent standard Gaussian white noises; $\{\sigma_k,\ k \geq 1\}$ is a CONS in H_C.

- σ_k^i is Hölder $\gamma/2$, $\sum_{k\geq 1} \sigma_k^i(x)\sigma_k^j(x) = \delta_{ij}$.
- Flow equation:

$$X^i(t;x,s) = x^i + \int_s^t \sigma_k^i(X(\tau;x,s))dw_k(\tau).$$

- *Statistical solution* of the flow equation.
- $\zeta = b/(a+b)$— degree of incompressibility.
  $\zeta = 0$— Pure Gradient;
  $\zeta = 1$— Incompressible.
- Classification of flows: phase diagram in (ζ,γ) plane.

Can be coalescing (particles stick together) or diffusive (particles split).

- Still very little info about θ.

Transport Equation as an SPDE

Notations: $D_i = \dfrac{\partial}{\partial x^i}$; $\Delta = D_i D_i$.

Summation convention: summation over a pair of repeating indices.

- $\mathbb{F} = (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$, a stochastic basis with the usual assumptions.
- $(w_k(t), k \geq 1, t \geq 0)$, independent standard Wiener processes on $\mathbb{F}$.
- $\mathbf{v}$ divergence-free (incompressible flow).

Then

$$d\theta(t,x) = -\sum_k \sigma_k(x) \cdot \nabla\theta(t,x) \circ dw_k(t).$$

or

$$\boxed{d\theta(t,x) = \frac{1}{2}\Delta\theta(t,x)dt - \sigma_k^i(x) D_i \theta(t,x) dw_k(t)}$$

Fully Degenerate Equation:

$$\frac{1}{2}\sigma_k^i(x)\sigma_k^j(x) = \frac{1}{2}\delta_{ij}.$$

Wiener Chaos

$W(t) = (w_k(t),\ k \geq 1,\ 0 \leq t \leq T)$.

$\{m_i(s),\ i \geq 1\}$ — CONS in $L_2([0,T])$,

$$\xi_{ik} = \int_0^T m_i(s)\,dw_k(s).$$

$$\mathcal{J} = \left\{\alpha = (\alpha_i^k,\ i,k \geq 1) \;\middle|\; |\alpha| = \sum_{i,k} \alpha_i^k < \infty\right\};$$

$$\xi_\alpha = \prod_{i,k} \left(\frac{H_{\alpha_i^k}(\xi_{ik})}{\sqrt{\alpha_i^k!}}\right), \text{ where}$$

$$H_n(x) = (-1)^n \exp\left\{\frac{x^2}{2}\right\} \frac{d^n}{dx^n} \exp\left\{-\frac{x^2}{2}\right\}.$$

Example: $\alpha = \begin{pmatrix} 0 & 1 & 0 & 3 & 0 & 0 & \cdots \\ 2 & 0 & 0 & 0 & 4 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots \end{pmatrix}$

$\alpha_2^1 = 1;\ \alpha_4^1 = 3;\ \alpha_1^2 = 2;\ \alpha_5^2 = 4.$

$$\xi_\alpha = \xi_{2,1} \cdot \frac{H_3(\xi_{4,1})}{\sqrt{3!}} \cdot \frac{H_2(\xi_{1,2})}{\sqrt{2!}} \cdot \frac{H_4(\xi_{5,2})}{\sqrt{4!}}.$$

Theorem. *(Cameron and Martin, 1947)*

The collection $\{\xi_\alpha,\ \alpha\ \in \mathcal{J}\}$ *is an orthonormal basis in* $L_2(\Omega, \mathcal{F}_T^W, \mathbb{P})$*:*

If $\eta \in L_2(\Omega, \mathcal{F}_T^W, \mathbb{P})$ *and* $\eta_\alpha = \mathbb{E}(\eta\xi_\alpha)$*, then*

$$\eta = \sum_{\alpha\in\mathcal{J}} \eta_\alpha \xi_\alpha$$

and

$$\mathbb{E}|\eta|^2 = \sum_{\alpha\in\mathcal{J}} \eta_\alpha^2.$$

Goal: $\theta(t,x) = \sum_{\alpha\in\mathcal{J}} \theta_\alpha(t,x)\xi_\alpha$

The S-system

$$d\theta(t,x) = \frac{1}{2}\Delta\theta(t,x)dt - \sigma^i_k(x)D_i\theta(t,x)dw_k(t)$$

If σ^i_k, θ_0 are smooth, then $\theta(t,x) = \sum\limits_{\alpha\in\mathcal{J}} \theta_\alpha(t,x)\xi_\alpha$.

Define: $\xi_\alpha(t) = \mathbb{E}(\xi_\alpha|\mathcal{F}^W_t)$; $\xi_\alpha(0) = I(|\alpha| = 0)$.

Fact: $d\xi_\alpha(t) = \sum\limits_{i,k} \sqrt{\alpha^k_i}\xi_{\alpha^-(i,k)}(t)m_i(t)dw_k(t)$,
where $\alpha^-(i,k)$ is the multi-index with the components

$$\Big(\alpha^-(i,k)\Big)^l_j = \begin{cases} \max(\alpha^k_i - 1, 0), & \text{if } i=j \text{ and } k=l, \\ \alpha^l_j, & \text{otherwise.} \end{cases}$$

By the Itô formula

$$\begin{aligned} \frac{\partial\theta_\alpha(t,x)}{\partial t} &= \frac{1}{2}\Delta\theta_\alpha(t,x) \\ &\quad - \sum_{i,k}\sqrt{\alpha^k_i}\sigma^j_k(x)D_j\theta_{\alpha^-(i,k)}(t,x)m_i(t); \\ \theta_\alpha(0,x) &= \theta_0(x)I(|\alpha| = 0) \end{aligned}$$

Solving the S-system

$|\alpha| = 0$:

$$\frac{\partial \theta_{(0)}(t,x)}{\partial t} = \frac{1}{2}\Delta \theta_{(0)}(t,x)$$

$\theta_{(0)}(0,x) = \theta_0(x) \;\Rightarrow\; \theta_{(0)}(t,x) = \mathbb{T}_t\theta_0(x).$

$\alpha = \delta_{ik}$:

$$\frac{\partial \theta_{(ik)}(t,x)}{\partial t} = \frac{1}{2}\Delta \theta_{(ik)}(t,x) - \sigma_k^j(x) D_j \theta_{(0)}(t,x) m_i(t);\;\; \theta_{(ik)}(0,x) = 0.$$

$$\theta_{(ik)}(t,x) = -\int_0^t m_i(s)\mathbb{T}_{t-s}\sigma_k^j D_j \mathbb{T}_s \theta_0(x) ds.$$

In fact, with $\mathcal{M}_k = -\sigma_k^j D_j$,

$$\sum_{|\alpha|=N} |\theta_\alpha(t,x)|^2 = \sum_{k_1,\ldots,k_N=1}^{\infty} \int_0^t \int_0^{s_N} \ldots \int_0^{s_2} |\mathbb{T}_{t-s_N}\mathcal{M}_{k_N} \ldots \mathbb{T}_{s_2-s_1}\mathcal{M}_{k_1}\mathbb{T}_{s_1}\theta_0(x)|^2 ds_1 \ldots ds_N.$$

Conclusion: σ_k^i do not have to be smooth or even continuous.

The Main Result

$d\theta(t,x) = \frac{1}{2}\Delta\theta(t,x)dt - \sigma_k^i(x)D_i\theta(t,x)dw_k(t)$

Theorem (Lototsky and Rozovskii, 2003)

If $\theta_0 \in L_2(\mathbb{R}^d)$, $D_i\sigma_k^i = 0$, and $\sigma_k^i(x)\sigma_k^j(x) = \delta_{ij}$, then

- For $t > 0$, $\sum_{\alpha\in\mathcal{J}} \|\theta_\alpha(t)\|^2_{L_2(\mathbb{R}^d)} \le \|u_0\|^2_{L_2(\mathbb{R}^d)}$.
- For every $\varphi \in \mathbf{C}_0^\infty(\mathbb{R}^d)$, the random field $\theta(t,x) = \sum_{\alpha\in\mathcal{J}} \theta_\alpha(t,x)\xi_\alpha$ satisfies

$$\begin{aligned}(\theta,\varphi)(t) &= (\theta_0,\varphi) + \frac{1}{2}\int_0^t (\theta,\Delta\varphi)(s)ds \\ &\quad + \int_0^t (\theta,\sigma_k^i D_i\varphi)dw_k(s).\end{aligned}$$

- If

$$X^i(s;t,x) = x^i - \int_s^t \sigma_k^i\left(X(s;t,x)\right)\overleftarrow{dw_k}(s)\,,\ s\in[0,t],$$

(weak solution), then

$$\theta\left(t,x\right) = \mathbb{E}\Big(\theta_0\left(X(0;t,x)\right)\big|\mathcal{F}_t^W\Big)$$

Discussion

- Transport Equation ⇒ The S-system ⇒ Coefficients $\theta_\alpha(t,x)$ ⇒

$$\text{Solution } \theta(t,x) = \sum_{\alpha\in\mathcal{J}} \theta_\alpha(t,x)\xi_\alpha.$$

- Probabilistically strong solution.

- $\mathbb{E}\theta(t,x) = \theta_{(0)}(t,x)$.

- $\mathbb{E}(\theta(t,x)\theta(s,y)) = \sum_{\alpha\in\mathcal{J}} \theta_\alpha(t,x)\theta_\alpha(s,y)$.

- By interpolation: $\mathbb{E}\|\theta\|^p_{L_p(\mathbb{R}^d)} \leq \|\theta_0\|^p_{L_p(\mathbb{R}^d)}$, $2 < p < \infty$. Weighted L_p (e.g. $\theta_0(x) = |x|$) are also OK.

- Incompressible flow ⇒ Itô = Stratonovich.

- Conservation of energy ⇔ Strong solution of the flow equation.

Totally Turbulent Transport

$$d\theta(t,x) = \nu\Delta\theta(t,x)dt - \sigma_k^i(x)D_i\theta(t,x)dw_k(t),$$

$\sigma_k,\ k \geq 1$ — CONS in $L_2(\mathbb{R}^d;\mathbb{R}^d)$.
(P. Chow, J. Potthoff, etc.)

Note: $\sum_{k\geq 1} \sigma_k^i(x)\sigma_k^j(x)$ diverges.

S-system:

$$\begin{aligned}\frac{\partial\theta_\alpha(t,x)}{\partial t} &= \nu\Delta\theta_\alpha(t,x)\\ &\quad - \sum_{i,k}\sqrt{\alpha_i^k}\sigma_k^j(x)D_j\theta_{\alpha^-(i,k)}(t,x)m_i(t);\end{aligned}$$

Still solvable, but now

$$\sum_{\alpha\in\mathcal{J}} \|\theta_\alpha(t)\|^2_{L_2(\mathbb{R}^d)} = +\infty.$$

Weighted Wiener Chaos

Idea: replace w_k with $q_k w_k$.

$Q = \{q_1, q_2, \ldots\}$, $q_k > 0$; $Q \le \tilde{Q}$ means $q_k \le \tilde{q}_k$ for all k;

$$q^{\alpha} = \prod_{i,k} q_k^{\alpha_i^k}.$$

Definition. *For a Banach space X, the Q-weighted Wiener Chaos space $L_{2,Q}(\mathcal{F}_T^W; X)$ is*

$$L_{2,Q}(\mathcal{F}_T^W; X) = \left\{ (u_\alpha) : \sum_{\alpha \in \mathcal{J}} q^{2\alpha} \|u_\alpha\|_X^2 < \infty \right\}.$$

Still write $u = \sum_{\alpha \in \mathcal{J}} u_\alpha \xi_\alpha$

$Q = 1 \Rightarrow L_{2,Q}(\mathcal{F}_T^W; X) = L_2(\mathcal{F}_T^W; X)$;
$Q \le \tilde{Q} \Rightarrow L_{2,\tilde{Q}}(\mathcal{F}_T^W; X) \subseteq L_{2,Q}(\mathcal{F}_T^W; X)$.

Q-transform:

$$u = \sum_{\alpha \in \mathcal{J}} u_\alpha \xi_\alpha \leftrightarrow u^Q = \sum_{\alpha \in \mathcal{J}} q^\alpha u_\alpha \xi_\alpha$$

$$du = \mathcal{A} u dt + \sum_{k \ge 1} \mathcal{B}_k u dw_k$$

$$\Updownarrow$$

$$du^Q = \mathcal{A} u^Q dt + \sum_{k \ge 1} q_k \mathcal{B}_k u^Q dw_k$$

Example: $w_k \longrightarrow q_k w_k$

1. (Obvious) If $u(t) = 1 + \sum_{k\geq 1} \int_0^t u(s)dw_k(s)$,

then $u \in L_{2,Q}(\mathcal{F}_T^W;\mathbb{R})$ for every $Q = (q_1, q_2, \ldots)$ so that

$$\sum_{k\geq 1} q_k^2 < \infty.$$

2. (Nualart-Rozovskii, 1997) If

$du(t,x) = \Delta u(t,x)dt + u(t,x)dw(t,x)$,

$t > 0,\ x \in \mathbb{R}^d, \quad d \geq 2,$

then $u \in L_{2,Q}(\mathcal{F}_T^W; L_2(\mathbb{R}^d))$ for some Q.

Theorem (Lototsky and Rozovskii, 2004)

Assume that $\theta_0 \in L_2(\mathbb{R}^d)$ and $|\sigma^i_k(x)| \le C_k$.

Let Q be a sequence with $q_k = \dfrac{\sqrt{\delta\nu}}{d2^k C_k}$ for some $0 < \delta < 2$. If

$$\begin{aligned}\frac{\partial \theta_\alpha(t,x)}{\partial t} &= \nu \Delta \theta_\alpha(t,x) \\ &\quad - \sum_{i,k} \sqrt{\alpha^k_i} \sigma^j_k(x) D_j \theta_{\alpha^-(i,k)}(t,x) m_i(t);\end{aligned}$$

then $\sum_{\alpha \in \mathcal{J}} q^{2\alpha} \|\theta_\alpha(t)\|^2_{L_2(\mathbb{R}^d)} < \infty$
and $\theta(t,x) = \sum_{\alpha \in \mathcal{J}} \theta_\alpha(t,x)\xi_\alpha$ satisfies

$$\theta \in L_{2,Q}\left(\mathcal{F}^W_T; \mathbf{C}((0,T); L_2(\mathbb{R}^d)\right).$$

This θ is called the *Wiener Chaos solution* of the totally turbulent transport equation $d\theta(t,x) = \nu\Delta\theta(t,x)dt - \sigma^i_k(x) D_i \theta(t,x) dw_k(t)$.

Wiener Chaos Approach

- Equation for $u = u(t)$ ⇒ The S-system ⇒ Coefficients $u_\alpha(t)$ ⇒

$$\text{Solution } u(t) = \sum_\alpha u_\alpha(t)\xi_\alpha.$$

- Probabilistically strong solution.

- Computable expressions for the solution and its moments from the S-system.

- New regularity results.

- Possibilities for generalization.

Further Directions

- Anticipating equations.

- Elliptic equations.

- Nonlinear equations.