A Random Change of Variables and Applications to the Stochastic Porous Medium Equation with Multiplicative Time Noise

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Motivation

$$du = Au\,dt + f(t)u\,dW(t);$$

$$u = v\exp\left(\int_0^t f(s)dW(s) - \frac{1}{2}\int_0^t f^2(s)ds\right);\; dv = Av\,dt.$$

Can we do something similar with

$$du = F(u)dt + f(t)u\,dW(t)?$$

If the talk becomes boring:

For what (non-random) functions f does the integral

$$\int_0^\infty \exp\left(\int_0^t f(s)dW(s) - \frac{1}{2}\int_0^t f^2(s)ds\right)dt$$

converge with probability one? In some L_p?

The Change of Variables: General

$(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$; $X = X(t)$ — “integrator”

$$v = v(t,x), \quad v_t = F(v, Dv, D^2v, \dots),\ t > 0,\ x \in G \subseteq \mathbb{R}^d,$$
$$u = u(t,x), \quad du = F(u, Du, D^2u, \dots)dt + u\, dX(t)$$

Theorem: Change of variables in a homogeneous equation

IF $F(\lambda x, \lambda y, \lambda z, \dots) = \lambda^\gamma F(x, y, z, \dots),\ \lambda > 0,\ \gamma > 0;$

$$h(t):\ dh(t) = h(t)\, dX(t),\ h(0) = 1;\ H_\gamma(t) = \int_0^t h^{\gamma-1}(s)ds.$$

THEN $u(t,x) = v(H_\gamma(t), x)h(t)$.
Proof. $du = v_t(H_\gamma(t), x)h^\gamma(t)dt + v(H_\gamma(t), x)h\, dX$
$F(u, Du, \dots) = F(hv, hDv, \dots) = h^\gamma F(v, Dv, \dots).$

Examples of X and h

- $dX(t) = f(t)dW(t) + g(t)dt$; Itô integral.

$$h(t) = \exp\left(\int_0^t g(s)ds + \int_0^t f(s)dW(s) - \frac{1}{2}\int_0^t f^2(s)ds\right).$$

- X — continuous semi-martingale; Itô integral.

$$h(t) = \exp\left(X(t) - X(0) - \frac{1}{2}\langle X^c\rangle_t\right).$$

- $X = W^H$ — fBM; Skorokhod integral.

$$h(t) = \exp\left(W^H(t) - \frac{1}{2}t^{2H}\right).$$

An Example of an Equation

$du = uu_x dt + udW(t)$: find a solution.

We have $v_t = vv_x$; $v(t,x) = -x/t$ works.

$\gamma = 2,\ h(t) = \exp\left(W(t) - \frac{1}{2}t\right)$.

Conclusion:

$$u(t,x) = -\frac{x \exp\left(W(t) - \frac{1}{2}t\right)}{\int_0^t \exp\left(W(s) - \frac{1}{2}s\right) ds}.$$

The Change of Variables: PME

$$v = v(t,x), \quad v_t = \mathbf{\Delta}(v^{\gamma}),\ t > 0,\ \gamma > 1;$$
$$u = u(t,x), \quad du = \mathbf{\Delta}(u^{\gamma})dt + u\,(f(t)dW(t) + g(t)dt)$$

Theorem. IF

$$h(t) = \exp\left(\int_0^t g(s)ds + \int_0^t f(s)dW(s) - \frac{1}{2}\int_0^t f^2(s)ds\right),$$
$$H_{\gamma}(t) = \int_0^t h^{\gamma-1}(s)ds,$$

THEN

$$u(t,x) = v(H_{\gamma}(t), x)h(t).$$

$f = 0,\ g = const.$: Gurtin, M. E., MacCamy, R. C. (1977) On the diffusion of biological populations. *Math. Biosci.*

Background on PME

A distributed system: density $\rho = \rho(t,x)$, velocity $\boldsymbol{v} = \boldsymbol{v}(t,x)$

Time evolution: $\sigma = \sigma(t,x,\rho)$ — density of sources and sinks, κ — fraction of the space available to the system;

equation of continuity $\kappa\rho_t + \mathrm{div}(\rho\,\boldsymbol{v}) = \sigma$

equation of motion $\boldsymbol{v} = \boldsymbol{F}(t,x,\rho,\mathbf{grad}\rho)$

Example Darcy's Law $\boldsymbol{v} = -c\,\mathbf{grad}\,p$, where $c > 0$ and $p = p_0\rho^\alpha$, $\alpha > 0$, is the pressure.

Assumption: $\boldsymbol{F}(t,x,\rho,\mathbf{grad}\,\rho) = -q(\rho)\mathbf{grad}\rho - \boldsymbol{b}\,\psi(\rho)$

Then $\boxed{\kappa\,\rho_t = \boldsymbol{\Delta}\Phi(\rho) + \boldsymbol{b}\cdot\mathbf{grad}\Psi(\rho) + \sigma(t,x,\rho)}$;

$\Phi(x) = \int_0^x yq(y)dy$, $\Psi(x) = x\psi(x)$.

$\boxed{\text{Rescaling: from } \rho \text{ to } u}$

Darcy's Law after re-scaling: $\Phi(x) = x^{1+\alpha}$, $\boldsymbol{b} = \mathbf{0}$. We now add reproduction $\sigma(t,x,u) = (g(t) + f(t)\dot{W}(t))u$ (random-in-time, crowd-avoiding population)

Some existing work

$u_t = \mathbf{\Delta}(u^\gamma) + \sigma(t,x,u)$, $\gamma > 1$.

(negative $u \Rightarrow$ use $|u|^{\gamma-1}u$ rather than u^γ)

- $\sigma(t,x,u) = u^\beta + u\dot{W}(t)$: Mel′nik (2001, 2002).
- $\sigma(t,x,u) = \sum_k f_k(t,x)\dot{W}_k(t)$: J. U. Kim (2006).
- $\sigma(t,x,u) = F(u) + \dot{W}(t,x)$: Barbu-Bogachev-Da Prato-Röckner (2006), and Da Prato-Röckner-Rozovskii-Wang (2006).
- $\sigma(t,x,u) = F(u) + G(u)\dot{W}(t,x)$: Barbu-Da Prato-Röckner (2007).
- Random $\boldsymbol{v}$: Sango (2007).

Deterministic theory:

- Aronson (1986, 50 pages).
- J. L. Vázquez (2007, 600 pages).

Solution of Stochastic PME

$v_t = \mathbf{\Delta}(v^\gamma),\ du = \mathbf{\Delta}(u^\gamma)dt + u\,(f(t)dW(t) + g(t)dt),\ u(t,x) = v(H_\gamma(t), x)h(t);\ x \in \mathbb{R}^d$

Definition A non-negative, continuous random field $u = u(t,x)$ is called a solution of the stochastic PME on the set $(\!(0,\tau]\!] = \{(t,\,\omega) : t \leq \tau\}$ if, for every smooth compactly supported function $\varphi = \varphi(x)$ the following equality holds for all $(t,\omega) \in (\!(0,\tau]\!]$:

$(u,\varphi)(t) =$

$$(u,\varphi)(0) + \int_0^t (u^\gamma, \mathbf{\Delta}\varphi)(s)ds + \int_0^t (u,\varphi)(s)(f(s)dW(s) + g(s)ds),$$

where $(u,\varphi)(t) = \displaystyle\int_{\mathbb{R}^d} u(t,x)\varphi(x)dx.$

Scaled pressure: $V(t,x) = \frac{\gamma}{\gamma-1}u^{\gamma-1}(t,x),$

$$dV = \left((\gamma-1)V\mathbf{\Delta}V + |\nabla V|^2 + \frac{(\gamma-1)(\gamma-2)}{2}Vf^2\right)dt$$

$+(\gamma-1)V\,(fdW + gdt).$

PME: from deterministic to stochastic

$v_t = \boldsymbol{\Delta}(v^\gamma),\ du = \boldsymbol{\Delta}(u^\gamma)dt + u\,(f(t)dW(t) + g(t)dt),\ u(t,x) = v(H_\gamma(t),x)h(t);\ x \in \mathbb{R}^d$

Theorem. Assume that the initial condition $u(0,x)$ is non-random, non-negative and bounded, continuous, integrable, and square-integrable.

Then there exists a unique non-negative solution $u = u(t,x)$, $t > 0$, $x \in \mathbb{R}^d$ with the following properties:

- u is Hölder continuous on $[T, +\infty) \times \mathbb{R}^d$ for every $T > 0$;
- if $\left(\int_{\mathbb{R}^d} u^p(0,x)dx\right)^{1/p} = M_p < \infty$, $p \geq 1$, then

$$\left(\mathbb{E}\int_{\mathbb{R}^d} u^p(t,x)dx\right)^{1/p} \leq M_p \exp\left(\int_0^t g(s)ds + \frac{(p-1)}{2}\int_0^t f^2(s)ds\right).$$

- If $g = 0$, then $\mathbb{E}\int_{\mathbb{R}^d} u(t,x)dx = M_1$ for all t.

PME: from deterministic to stochastic

$v_t = \mathbf{\Delta}(v^\gamma),\ du = \mathbf{\Delta}(u^\gamma)dt + u\left(f(t)dW(t) + g(t)dt\right),\ u(t,x) = v(H_\gamma(t),x)h(t);\ x \in \mathbb{R}^d$

$h(t) = \exp\left(\int_0^t g(s)ds + \int_0^t f(s)dW(s) - \frac{1}{2}\int_0^t f^2(s)ds\right), H_\gamma(t) = \int_0^t h^{\gamma-1}(s)ds$

Comparison principle: If $0 \leq u(0,x) \leq \widetilde{u}(0,x)$ for all $x \in \mathbb{R}^d$, then $u(t,x) \leq \widetilde{u}(t,x)$.

Maximum principle: If $0 < m \leq u(0,x) \leq M$ for all $x \in \mathbb{R}^d$, then $mh(t) \leq u(t,x) \leq Mh(t)$.

Possible blow-up: $u^{[\mathrm{qp}]}(t,x) = \left(\frac{t_1|x|^2}{t_q - H_\gamma(t)}\right)^{1/(\gamma-1)} h(t),$

$$t_q = \frac{\gamma-1}{2\gamma q(2+d(\gamma-1))},\ q > 0;\ t_1 = t_q|_{q=1}.$$

No blow-up with unbounded initial condition:

$$u^{[\mathrm{lp}]}(t,x) = \left(\frac{\gamma-1}{\gamma}\max\left(H_\gamma(t)+x,\ 0\right)\right)^{1/(\gamma-1)} h(t).$$

Long-Time Behavior, Part 1

$v_t = \mathbf{\Delta}(v^\gamma),\ du = \mathbf{\Delta}(u^\gamma)dt + u\,(f(t)dW(t) + g(t)dt),\ u(t,x) = v(H_\gamma(t),x)h(t);\ x \in \mathbb{R}^d$

Barenblatt's family of solutions:

$u^{[\mathrm{BT}]}(t,x;b) = U^{[\mathrm{BT}]}(H_\gamma(t),x;b)h(t)$, where

$$U^{[\mathrm{BT}]}(t,x;b) = \frac{1}{t^\alpha}\left(\max\left(b - \frac{\gamma-1}{2\gamma}\,\beta\,\frac{|x|^2}{t^{2\beta}},\ 0\right)\right)^{1/(\gamma-1)},$$

$$b > 0,\ \beta = \frac{1}{(\gamma-1)d+2},\ \ \alpha = \beta d.$$

Theorem. Assume that $\lim_{t\to\infty} H_\gamma(t) = +\infty$ with probability one. Then, for every $x \in \mathbb{R}^d$, $\lim\limits_{t\to\infty}(H_\gamma(t))^{\beta d}|u(t,x) - u^{[\mathrm{BT}]}(t,x;b)| = 0$ with probability one, where b is such that

$$\int_{\mathbb{R}^d} u(0,x)dx = b^{1/(2\beta(\gamma-1))}\left(\frac{\gamma-1}{2\pi\,\gamma}\,\beta\right)^{-d/2}\frac{\Gamma\left(\frac{\gamma}{\gamma-1}\right)}{\Gamma\left(\frac{\gamma}{\gamma-1}+\frac{d}{2}\right)}.$$

Long-Time Behavior, Part 2

$v_t = \mathbf{\Delta}(v^\gamma),\ du = \mathbf{\Delta}(u^\gamma)dt + u\,(f(t)dW(t) + g(t)dt),\ u(t,x) = v(H_\gamma(t),x)h(t);\ x \in \mathbb{R}^d$

$h(t) = \exp\left(\int_0^t g(s)ds + \int_0^t f(s)dW(s) - \frac{1}{2}\int_0^t f^2(s)ds\right), H_\gamma(t) = \int_0^t h^{\gamma-1}(s)ds$

What if $\lim\limits_{t\to\infty} h(t) > 0$ exists with probability one?

Then $\lim\limits_{t\to\infty} H_\gamma(t) = +\infty$ for sure.

Theorem. Assume that $\int_0^\infty f^2(t)dt = 2\sigma^2$ and $\int_0^\infty g(t)dt = \mu$ for some $\sigma, \mu \in \mathbb{R}$. Then, for every $x \in \mathbb{R}^d$,

$$\lim_{t\to\infty} |u(t,x) - e^\xi\, U^{[\mathrm{BT}]}(e^{(\gamma-1)\xi}t, x; b)| = 0$$

with probability one and the same b, where ξ is a Gaussian random variable with mean $\mu - \sigma^2$ and variance σ^2.

Long-Time Behavior, Part 3

$v_t = \boldsymbol{\Delta}(v^\gamma),\ du = \boldsymbol{\Delta}(u^\gamma)dt + u\left(f(t)dW(t) + g(t)dt\right),\ u(t,x) = v(H_\gamma(t), x)h(t);\ x \in \mathbb{R}^d$

$h(t) = \exp\left(\int_0^t g(s)ds + \int_0^t f(s)dW(s) - \frac{1}{2}\int_0^t f^2(s)ds\right), H_\gamma(t) = \int_0^t h^{\gamma-1}(s)ds$

Theorem. Assume that the initial condition $u(0,x)$ is continuous, non-negative, and compactly supported in $\mathbb{R}^d$. Then

- the solution $u(t,x)$ is non-negative and has compact support in $\mathbb{R}^d$ for all $t > 0$;
- the interface, that is, the boundary of the set $\{x \in \mathbb{R}^d : u(t,x) > 0\}$, is moving with finite speed.
- If $\lim_{t\to\infty} H(t) < \infty$, then the support of the solution remains bounded for all $t > 0$.

An Example

$du = \boldsymbol{\Delta}(u^2)\,dt + u\,dW(t)\ \ (\gamma = 2)$, $u(0,x)$ is continuous, non-negative, compactly supported, and $\int_{\mathbb{R}^d} u(0,x)dx > 0$. There exists a random variable η, $0 < \eta < \infty$ with probability one, $u(t,x) = 0$, $|x| > \eta$, $t > 0$. Indeed

$$h(t) = e^{w(t)-(t/2)},\ H_\gamma(t) = \int_0^t e^{w(s)-(s/2)}ds,$$

$H_\gamma(t)$ is bounded (by the Law of Iterated Logarithm).
In particular $\lim_{t\to\infty} h(t) = 0$, so that, for every $x \in \mathbb{R}^d$, $\lim_{t\to\infty} u(t,x) = 0$. On the other hand,
$\mathbb{E}\displaystyle\int_{\mathbb{R}^d} u(t,x)dx = \int_{\mathbb{R}^d} u(0,x)dx$: **the solution is supported in the same (random) compact set for all $t > 0$ and decays to zero as $t \to \infty$, while preserving the expected total mass.**

Note: the support of $v_t = \boldsymbol{\Delta}(v^2)$, $v(0,x) = u(0,x)$, is not bounded.

So...

How much of the above can we do for

$$du = \boldsymbol{\Delta}(u^\gamma)dt + \sum_{k\geq 1} f_k(t,x)dW_k(t)?$$