A Random Change of Variables and Applications to the Stochastic Porous Medium Equation with Multiplicative Time Noise

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\[ du = Au \, dt + f(t)u \, dW(t); \]
\[ u = v \exp \left( \int_{0}^{t} f(s) \, dW(s) - \frac{1}{2} \int_{0}^{t} f^2(s) \, ds \right) ; \quad dv = Av \, dt. \]

Can we do something similar with
\[ du = F(u) \, dt + f(t)u \, dW(t)? \]

If the talk becomes boring:
For what (non-random) functions \( f \) does the integral
\[ \int_{0}^{\infty} \exp \left( \int_{0}^{t} f(s) \, dW(s) - \frac{1}{2} \int_{0}^{t} f^2(s) \, ds \right) \, dt \]
converge with probability one? In some \( L_p \)?
The Change of Variables: General

\( (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P}); \ X = X(t) \) — “integrator”

\[ v = v(t, x), \quad v_t = F(v, Dv, D^2v, \ldots), \ t > 0, \ x \in G \subseteq \mathbb{R}^d, \]
\[ u = u(t, x), \quad du = F(u, Du, D^2u, \ldots)dt + u \, dX(t) \]

**Theorem:** Change of variables in a homogeneous equation

\[ F(\lambda x, \lambda y, \lambda z, \ldots) = \lambda^\gamma F(x, y, z, \ldots), \ \lambda > 0, \ \gamma > 0; \]
\[ h(t): \ dh(t) = h(t) \, dX(t), \ h(0) = 1; \ H_\gamma(t) = \int_0^t h_\gamma^{-1}(s)ds. \]

**THEN** \( u(t, x) = v(H_\gamma(t), x)h(t). \)

**Proof.** \( du = v_t(H_\gamma(t), x)h_\gamma(t)dt + v(H_\gamma(t), x)h \, dX \)
\[ F(u, Du, \ldots) = F(hv, hDv, \ldots) = h^\gamma F(v, Dv, \ldots). \]
Examples of $X$ and $h$

- $dX(t) = f(t)dW(t) + g(t)dt$; Itô integral.
  \[ h(t) = \exp \left( \int_0^t g(s)ds + \int_0^t f(s)dW(s) - \frac{1}{2} \int_0^t f^2(s)ds \right). \]

- $X$ — continuous semi-martingale; Itô integral.
  \[ h(t) = \exp \left( X(t) - X(0) - \frac{1}{2} \langle X^c \rangle_t \right). \]

- $X = W^H$ — fBM; Skorokhod integral.
  \[ h(t) = \exp \left( W^H(t) - \frac{1}{2} t^{2H} \right). \]
du = uu_x dt + udW(t) : find a solution.

We have \( v_t = vv_x ; v(t, x) = -x/t \) works.

\( \gamma = 2, \ h(t) = \exp \left( W(t) - \frac{1}{2}t \right) \).

Conclusion:

\[
 u(t, x) = -\frac{x \exp \left( W(t) - \frac{1}{2}t \right)}{\int_0^t \exp \left( W(s) - \frac{1}{2}s \right) ds}.
\]
\[ v = v(t, x), \quad v_t = \Delta(v^\gamma), \quad t > 0, \quad \gamma > 1; \]
\[ u = u(t, x), \quad du = \Delta(u^\gamma)dt + u(f(t)dW(t) + g(t)dt) \]

**Theorem.**  IF

\[
h(t) = \exp \left( \int_0^t g(s)ds + \int_0^t f(s)dW(s) - \frac{1}{2} \int_0^t f^2(s)ds \right),
\]

\[
H_\gamma(t) = \int_0^t h^{\gamma-1}(s)ds,
\]

THEN

\[
u(t, x) = v(H_\gamma(t), x)h(t).
\]

\[ f = 0, \quad g = const.: \] Gurtin, M. E., MacCamy, R. C. (1977)

A distributed system: density $\rho = \rho(t, x)$, velocity $\bm{v} = \bm{v}(t, x)$

**Time evolution:** $\sigma = \sigma(t, x, \rho)$ — density of sources and sinks, $\kappa$ — fraction of the space available to the system;

**equation of continuity** $\kappa \rho_t + \text{div}(\rho \bm{v}) = \sigma$

**equation of motion** $\bm{v} = F(t, x, \rho, \text{grad}\rho)$

**Example** Darcy’s Law $\bm{v} = -c \text{grad}\rho$, where $c > 0$ and $p = p_0 \rho^\alpha$, $\alpha > 0$, is the pressure.

**Assumption:** $F(t, x, \rho, \text{grad}\rho) = -q(\rho)\text{grad}\rho - b \psi(\rho)$

Then $\kappa \rho_t = \Delta \Phi(\rho) + b \cdot \text{grad}\Psi(\rho) + \sigma(t, x, \rho)$;

$\Phi(x) = \int_0^x yq(y)dy$, $\Psi(x) = x\psi(x)$.

**Rescaling:** from $\rho$ to $u$

Darcy’s Law after re-scaling: $\Phi(x) = x^{1+\alpha}$, $b = 0$. We now add reproduction $\sigma(t, x, u) = (g(t) + f(t)\dot{W}(t))u$ (random-in-time, crowd-avoiding population)
Some existing work

\[ u_t = \Delta (u^\gamma) + \sigma(t, x, u), \quad \gamma > 1. \]

(negative \( u \) \( \Rightarrow \) use \( |u|^{\gamma-1} u \) rather than \( u^\gamma \))

- \( \sigma(t, x, u) = u^\beta + u \dot{W}(t) \): Mel’nik (2001, 2002).
- \( \sigma(t, x, u) = \sum_k f_k(t, x) \dot{W}_k(t) \): J. U. Kim (2006).
- \( \sigma(t, x, u) = F(u) + \dot{W}(t, x) \): Barbu-Bogachev-Da Prato-Röckner (2006), and Da Prato-Röckner-Rozovskii-Wang (2006).
- \( \sigma(t, x, u) = F(u) + G(u) \dot{W}(t, x) \): Barbu-Da Prato-Röckner (2007).
- Random \( \nu \): Sango (2007).

Deterministic theory:
- J. L. Vázquez (2007, 600 pages).
Solution of Stochastic PME

\[ v_t = \Delta (v^\gamma), \quad du = \Delta (u^\gamma)dt + u (f(t)dW(t) + g(t)dt), \quad u(t, x) = v(H^\gamma(t), x)h(t); \quad x \in \mathbb{R}^d \]

**Definition** A non-negative, continuous random field \( u = u(t, x) \) is called a solution of the stochastic PME on the set \( (0, \tau] = \{(t, \omega) : t \leq \tau\} \) if, for every smooth compactly supported function \( \varphi = \varphi(x) \) the following equality holds for all \( (t, \omega) \in (0, \tau] \):

\[
(u, \varphi)(t) = (u, \varphi)(0) + \int_0^t (u^\gamma, \Delta \varphi)(s)ds + \int_0^t (u, \varphi)(s)(f(s)dW(s) + g(s)ds),
\]

where \( (u, \varphi)(t) = \int_{\mathbb{R}^d} u(t, x)\varphi(x)dx \).

**Scaled pressure:** \( V(t, x) = \frac{\gamma}{\gamma - 1} u^{\gamma - 1}(t, x), \)

\[
dV = \left( (\gamma - 1)V \Delta V + |\nabla V|^2 + \frac{(\gamma - 1)(\gamma - 2)}{2} V f^2 \right) dt + (\gamma - 1)V (f dW + g dt).
\]
\[ vt = \Delta (v^\gamma), \quad du = \Delta (u^\gamma) dt + u (f(t)dW(t) + g(t)dt), \quad u(t, x) = v(H^\gamma(t), x)h(t); \quad x \in \mathbb{R}^d \]

**Theorem.** Assume that the initial condition \( u(0, x) \) is non-random, non-negative and bounded, continuous, integrable, and square-integrable.

Then there exists a unique non-negative solution \( u = u(t, x), \; t > 0, \; x \in \mathbb{R}^d \) with the following properties:

- \( u \) is Hölder continuous on \([T, +\infty) \times \mathbb{R}^d\) for every \( T > 0 \);
- if \( \left( \int_{\mathbb{R}^d} u^p(0, x)dx \right)^{1/p} = M_p < \infty, \; p \geq 1 \), then
  \[
  \left( \mathbb{E} \int_{\mathbb{R}^d} u^p(t, x)dx \right)^{1/p} \leq M_p \exp \left( \int_0^t g(s)ds + \frac{(p - 1)}{2} \int_0^t f^2(s)ds \right).
  \]
- If \( g = 0 \), then \( \mathbb{E} \int_{\mathbb{R}^d} u(t, x)dx = M_1 \) for all \( t \).
\[ v_t = \Delta (v^\gamma), \quad du = \Delta (u^\gamma) dt + u (f(t)dW(t) + g(t) dt), \quad u(t, x) = v(H_\gamma(t), x) h(t); \quad x \in \mathbb{R}^d \]

\[ h(t) = \exp \left( \int_0^t g(s) ds + \int_0^t f(s) dW(s) - \frac{1}{2} \int_0^t f^2(s) ds \right), \quad H_\gamma(t) = \int_0^t h^{\gamma-1}(s) ds \]

**Comparison principle:** If \( 0 \leq u(0, x) \leq \tilde{u}(0, x) \) for all \( x \in \mathbb{R}^d \), then \( u(t, x) \leq \tilde{u}(t, x) \).

**Maximum principle:** If \( 0 < m \leq u(0, x) \leq M \) for all \( x \in \mathbb{R}^d \), then \( mh(t) \leq u(t, x) \leq Mh(t) \).

**Possible blow-up:** \( u^{[qp]}(t, x) = \left( \frac{t_1 |x|^2}{t_q - H_\gamma(t)} \right)^{1/(\gamma-1)} h(t), \)

\[ t_q = \frac{\gamma - 1}{2\gamma q(2 + d(\gamma - 1))}, \quad q > 0; \quad t_1 = t_q|_{q=1}. \]

**No blow-up with unbounded initial condition:** \( u^{[lp]}(t, x) = \left( \frac{\gamma - 1}{\gamma} \max (H_\gamma(t) + x, 0) \right)^{1/(\gamma-1)} h(t). \)
\[ v_t = \Delta (v^\gamma), \quad du = \Delta (u^\gamma) dt + u (f(t)dW(t) + g(t)dt), \quad u(t, x) = v(H_\gamma(t), x)h(t); \quad x \in \mathbb{R}^d \]

Barenblatt's family of solutions:
\[ u^{[BT]}(t, x; b) = U^{[BT]}(H_\gamma(t), x; b)h(t), \text{ where} \]
\[ U^{[BT]}(t, x; b) = \frac{1}{t^\alpha} \left( \max \left( b - \frac{\gamma - 1}{2\gamma} \beta \frac{|x|^2}{t^{2\beta}}, 0 \right) \right)^{1/(\gamma-1)}, \]
\[ b > 0, \quad \beta = \frac{1}{(\gamma - 1)d + 2}, \quad \alpha = \beta d. \]

**Theorem.** Assume that \( \lim_{t \to \infty} H_\gamma(t) = +\infty \) with probability one. Then, for every \( x \in \mathbb{R}^d, \) \( \lim_{t \to \infty} (H_\gamma(t))^{\beta d}|u(t, x) - u^{[BT]}(t, x; b)| = 0 \) with probability one, where \( b \) is such that
\[
\int_{\mathbb{R}^d} u(0, x)dx = b^{1/(2\beta(\gamma-1))} \left( \frac{\gamma - 1}{2\pi \gamma} \beta \right)^{-d/2} \frac{\Gamma \left( \frac{\gamma}{\gamma-1} \right)}{\Gamma \left( \frac{\gamma}{\gamma-1} + \frac{d}{2} \right)}.
\]
\[ v_t = \Delta(v^\gamma), \quad du = \Delta(u^\gamma)dt + u(f(t)dW(t) + g(t)dt), \quad u(t, x) = v(H_\gamma(t), x)h(t); \quad x \in \mathbb{R}^d \]

\[ h(t) = \exp \left( \int_0^t g(s)ds + \int_0^t f(s)dW(s) - \frac{1}{2} \int_0^t f^2(s)ds \right), \quad H_\gamma(t) = \int_0^t h^\gamma^{-1}(s)ds \]

What if \( \lim_{t \to \infty} h(t) > 0 \) exists with probability one?

Then \( \lim_{t \to \infty} H_\gamma(t) = +\infty \) for sure.

**Theorem.** Assume that \( \int_0^\infty f^2(t)dt = 2\sigma^2 \) and \( \int_0^\infty g(t)dt = \mu \) for some \( \sigma, \mu \in \mathbb{R} \). Then, for every \( x \in \mathbb{R}^d \),

\[ \lim_{t \to \infty} |u(t, x) - e^\xi U^{[BT]}(e^{(\gamma-1)\xi}t, x; b)| = 0 \]

with probability one and the same \( b \), where \( \xi \) is a Gaussian random variable with mean \( \mu - \sigma^2 \) and variance \( \sigma^2 \).
\[ v_t = \Delta (v^\gamma), \quad du = \Delta (u^\gamma) dt + u (f(t)dW(t) + g(t)dt), \quad u(t, x) = v(H_\gamma(t), x)h(t); \quad x \in \mathbb{R}^d \]

\[ h(t) = \exp \left( \int_0^t g(s)ds + \int_0^t f(s)dW(s) - \frac{1}{2} \int_0^t f^2(s)ds \right), \quad H_\gamma(t) = \int_0^t h^{\gamma-1}(s)ds \]

**Theorem.** Assume that the initial condition \( u(0, x) \) is continuous, non-negative, and compactly supported in \( \mathbb{R}^d \). Then

- the solution \( u(t, x) \) is non-negative and has compact support in \( \mathbb{R}^d \) for all \( t > 0 \);
- the interface, that is, the boundary of the set \( \{ x \in \mathbb{R}^d : u(t, x) > 0 \} \), is moving with finite speed.
- If \( \lim_{t \to \infty} H(t) < \infty \), then the support of the solution remains bounded for all \( t > 0 \).
$d u = \Delta (u^2) \, dt + u \, dW(t) \quad (\gamma = 2), \quad u(0, x)$ is continuous, non-negative, compactly supported, and $\int_{\mathbb{R}^d} u(0, x) \, dx > 0$. There exists a random variable $\eta$, $0 < \eta < \infty$ with probability one, $u(t, x) = 0$, $|x| > \eta$, $t > 0$. Indeed

$$h(t) = e^{w(t) - (t/2)}, \quad H_\gamma(t) = \int_0^t e^{w(s) - (s/2)} \, ds,$$

$H_\gamma(t)$ is bounded (by the Law of Iterated Logarithm).

In particular $\lim_{t \to \infty} h(t) = 0$, so that, for every $x \in \mathbb{R}^d$, $\lim_{t \to \infty} u(t, x) = 0$. On the other hand,

$$\mathbb{E} \int_{\mathbb{R}^d} u(t, x) \, dx = \int_{\mathbb{R}^d} u(0, x) \, dx: \text{the solution is supported in the same (random) compact set for all } t > 0 \text{ and decays to zero as } t \to \infty, \text{ while preserving the expected total mass.}$$

**Note:** the support of $v_t = \Delta (v^2)$, $v(0, x) = u(0, x)$, is not bounded.
How much of the above can we do for

\[ du = \Delta (u^\gamma) dt + \sum_{k \geq 1} f_k(t, x) dW_k(t) \]