

Limit of a sequence

PROBLEMS.

- (1) (65A5) Show that, for a real sequence $\{a_k, k \geq 1\}$,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n e^{ia_k} = a \Leftrightarrow \lim_{n \rightarrow \infty} \frac{1}{n^2} \sum_{k=1}^n e^{ia_{k^2}} = a.$$

($i = \sqrt{-1}$).

- (2) (66A3) For $0 < x_1 < 1$ and $x_{n+1} = x_n(1 - x_n)$, show that $\lim_{n \rightarrow +\infty} nx_n = 1$.
(3) (66A6) Show that

$$3 = \sqrt{1 + 2\sqrt{1 + 3\sqrt{1 + 4\sqrt{1 + 5\sqrt{1 + \dots}}}}}$$

- (4) (69A6) For a given sequence $\{x_n, n \geq 1\}$, assume that $\lim_{n \rightarrow \infty} (2x_n + x_{n-1})$ exists. Show that $\lim_{n \rightarrow \infty} x_n$ exists.
(5) (70A4) For a given sequence $\{x_n, n \geq 1\}$, assume that $\lim_{n \rightarrow \infty} (x_n - x_{n-2}) = 0$. Show that $\lim_{n \rightarrow \infty} \frac{x_n - x_{n-1}}{n} = 0$.
(6) (82B5) For $x > e^e$, define $u_0 = e$ and $u_{n+1} = \log_{u_n} x$. Show that $g = g(x) = \lim_{n \rightarrow \infty} u_n$ exists and the function $x \mapsto g(x)$ is continuous.
(7) (82B3) Two numbers are chosen at random from the set $(1, \dots, n)$ according to the uniform distribution. Let p_n be the probability that the sum of the numbers is a perfect square. Find $\lim_{n \rightarrow \infty} \sqrt{n}p_n$.
(8) (87B4) Define $x_1 = 0.8$, $y_1 = 0.6$ and $x_{n+1} = x_n \cos y_n - y_n \sin y_n$, $y_n = x_n \sin y_n + y_n \cos y_n$. Find $\lim_{n \rightarrow \infty} x_n$ and $\lim_{n \rightarrow \infty} y_n$.