Computations

Problems.

(1) Using that

$$\frac{\pi}{2} = \frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdot\frac{6}{5}\cdot\frac{6}{7}\cdot\frac{8}{7}\cdots\frac{2m\cdot 2m}{(2m-1)(2m+1)}\cdots$$

(the Wallis product),

- (69B3) Find a_1 if $a_n a_{n+1} = n$ and $\lim_{n\to\infty}\frac{a_n}{a_{n+1}} = 1$.
- (83B5) Find $\lim_{n\to\infty} a_n$, where

$$a_n = \frac{1}{n}\int_1^n \left\|\frac{n}{x}\right\| dx,$$

and $\|x\|$ is the distance from x to the nearest integer (e.g. $\|3.1\| = \|3.9\| = \|2.1\| = 0.1$.)

(2) (89A2) Compute

$$\int_0^a \int_0^b e^{\max(b^2x^2, a^2y^2)} dy dx,\ \ a, b > 0.$$

(3) (89B3) Assume that $|f(x)| \leq e^{-\sqrt{x}}$, $x \geq 0$, and

$$f'(x) = -3f(x) + 6f(2x), x > 0.$$

Define $\mu_n = \int_0^\infty x^n f(x)dx$. Find the expression for μ_n in terms of μ_0, show that that $a = \lim_{n\to\infty}\mu_n 3^n/n!$ exists and $a = 0$ if and only if $\mu_0 = 0$.

(4) For what real x does the series

$$\sum_{n\geq 0}\left(\frac{1}{n}\csc\frac{1}{n} - 1\right)^x$$

converge?

(5) (87B1) Evaluate

$$\int_2^4 \frac{\sqrt{\ln(9-x)}\ dx}{\sqrt{\ln(9-x)} + \sqrt{\ln(x+3)}}$$

(This is very easy as long as you do not try to find an antiderivative).

(6) (86A3) Evaluate

$$\sum_{n\geq 0} arccot(n^2+n+1),$$

where, for $x \geq 0$, $\theta = arccot(x)$ is the number from $(0, \pi/2]$ so that $\cot(\theta) = x$.

(7) (77B1) Evaluate

$$\prod_{n\geq 1}\frac{n^3-1}{n^3+1}.$$