TIME EVOLUTION OF A PASSIVE SCALAR IN A TURBULENT INCOMPRESSIBLE GAUSSIAN VELOCITY FIELD

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ABSTRACT. Passive scalar equation is considered in a turbulent homogeneous incompressible Gaussian velocity field. The turbulent nature of the field results in non-smooth coefficients in the equation. A strong, in the stochastic sense, solution of the equation is constructed using the Wiener Chaos, and the properties of the solution are studied. The results apply to both viscous and conservative motions.

1. Passive Scalar in a Gaussian Field

We consider the following transport equation to describe the evolution of a passive scalar \( \theta \) in a random velocity field \( \mathbf{v} \):

\[
\dot{\theta}(t, x) = 0.5 \nu \Delta \theta(t, x) - \mathbf{v}(t, x) \cdot \nabla \theta(t, x) + f(t, x); \quad x \in \mathbb{R}^d, \quad d > 1.
\]

Our interest in this equation is motivated by the on-going progress in the study of the turbulent transport problem (E and Vanden Eijnden [3], Gawędzki and Kupiainen [4], Gawędzki and Vergasola [5], Kraichnan [7], etc.) We assume in (1.1) that \( \mathbf{v} = \mathbf{v}(t, x) \in \mathbb{R}^d, \ d \geq 2, \) is an isotropic Gaussian vector field with zero mean and covariance

\[
E(\mathbf{v}(t, x) \mathbf{v}(s, y)) = \delta(t-s)C^{ij}(x-y)
\]

with some matrix-valued function \( C = (C^{ij}(x), i, j = 1, \ldots, d) \). It is well-known (see, for example, LeJan [8]) that in the physically interesting models the matrix-valued function \( C = C(x) \) has the Fourier transform \( \hat{C} = \hat{C}(z) \) given by

\[
\hat{C}(z) = \frac{A_0}{(1 + |z|^2)^(d+\alpha)/2} \left( \frac{zz^*}{|z|^2} + \frac{b}{d-1} \left( I - \frac{zz^T}{|z|^2} \right) \right),
\]

where \( z^* \) is the row vector \((z_1, \ldots, z_d)\), \( z \) is the corresponding column vector, \( |z|^2 = z^*z \), \( I \) is the identity matrix; \( \alpha > 0, a \geq 0, b \geq 0, A_0 > 0 \) are real numbers. Similar to [8], we assume that \( 0 < \alpha < 2 \).

By direct computation (cf. [1]), the vector field \( \mathbf{v} = (v^1, \ldots, v^d) \) can be written as

\[
v^i(t, x) = \sum_{k \geq 0} \sigma^i_k(x) \tilde{w}_k(t),
\]

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where \( \dot{w}_k(t) \), \( k \geq 1 \), are independent standard Gaussian white noises and \( \{ \sigma_k, ~ k \geq 1 \} \) is a CONS in the space \( H_C \), the reproducing kernel Hilbert space corresponding to the kernel function \( C \). The space \( H_C \) is all or part of the Sobolev space \( H^{(d+\alpha)/2}(\mathbb{R}^d; \mathbb{R}^d) \). It follows from (1.2) that \( \sum_k \sigma_k^i(x)\sigma_k^j(y) = C^{ij}(x - y) \) for all \( x, y \); in particular, \( \sigma_k^i(x)\sigma_k^j(x) = C^{ij}(0) \) for all \( x \).

If \( a > 0 \) and \( b > 0 \), then the matrix \( \hat{C} \) is invertible and

\[
H_C = \{ f \in \mathbb{R}^d : \int_{\mathbb{R}^d} |\hat{f}(z)|^2 (1 + |z|^2)^{d+\alpha}/2 \, dz < \infty \} = H^{(d+\alpha)/2}(\mathbb{R}^d; \mathbb{R}^d),
\]

because \( \| \hat{C}(z) \| \sim (1 + |z|^2)^{-(d+\alpha)/2} \).

If \( a > 0 \) and \( b = 0 \), then

\[
H_C = \left\{ f \in \mathbb{R}^d : \int_{\mathbb{R}^d} |\hat{f}(z)|^2 (1 + |z|^2)^{d+\alpha}/2 \, dz < \infty; \ z \hat{f}(z) = |z|^2 \hat{f}(z) \right\},
\]

the subset of gradient fields in \( H^{(d+\alpha)/2}(\mathbb{R}^d; \mathbb{R}^d) \) (those are vector fields \( f \) for which \( \hat{f}(z) = z \hat{F}(z) \) for some scalar \( F \in H^{(d+\alpha+1)/2} \)).

If \( a = 0 \) and \( b > 0 \), then

\[
H_C = \left\{ f \in \mathbb{R}^d : \int_{\mathbb{R}^d} |\hat{f}(z)|^2 (1 + |z|^2)^{d+\alpha}/2 \, dz < \infty; \ z^* \hat{f}(z) = 0 \right\},
\]

the subset of divergence free fields in \( H^{(d+\alpha)/2}(\mathbb{R}^d; \mathbb{R}^d) \).

By the embedding theorems, each \( \sigma_k^i \) is a bounded continuous function on \( \mathbb{R}^d \); in fact, every \( \sigma_k^i \) is Hölder continuous of order \( \alpha/2 \). In addition, being an element of the corresponding space \( H_C \), each \( \sigma_k \) is a gradient field if \( b = 0 \) and is divergence free if \( a = 0 \).

To simplify the further presentation and to make the model (1.1) more physically relevant, we consider the divergence-free velocity field and assume that the stochastic integration is in the sense of Stratonovich. Under these assumptions, equation (1.1) becomes

\[
d\theta(t, x) = 0.5
\nu \Delta \theta(t, x) dt - \sum_k \sigma_k(x) \cdot \nabla \theta(t, x) \circ dw_k(t).
\]

With divergence-free functions \( \sigma_k \), the equivalent Ito formulation is

\[
d\theta(t, x) = 0.5(\nu \Delta \theta(t, x) + C^{ij}(0) D_i D_j \theta(t, x)) dt - \sigma_k^i(x) D_i \theta(t, x) dw_k(t).
\]

In what follows, we construct a solution of (1.4) using Wiener Chaos.

## 2. A Review of the Wiener Chaos

Let \( \mathbb{F} = (\Omega, \mathcal{F}, \{ \mathcal{F}_t \}_{t \geq 0}, \mathbb{P}) \) be a stochastic basis with the usual assumptions. On \( \mathbb{F} \) consider a collection \( \{ w_k(t), k \geq 1, t \geq 0 \} \) of independent standard Wiener processes. For a fixed \( 0 < T < \infty \), let \( \mathcal{F}^W_T \) be the sigma-algebra generated by \( w_k(t), k \geq 1, 0 < t < T \), and \( L_2(\mathcal{F}^W_T) \) the collection of \( \mathcal{F}^W_T \)-measurable square integrable random variables.
For the Fourier cosine basis \( \{ m_k, k \geq 1 \} \) in \( L_2((0, T)) \) with

\[
m_1(t) = \frac{1}{\sqrt{T}}, \quad m_k(t) = \sqrt{\frac{2}{T}} \cos \left( \frac{\pi(k-1)t}{T} \right), \quad k \geq 2,
\]
define the independent standard Gaussian random variables

\[
\xi_{ik} = \int_0^T m_i(s)dw_k(s).
\]

Consider the collection of multi-indices

\[
\mathcal{J} = \left\{ \alpha = (\alpha_i^k, i, k \geq 1), \quad \alpha_i^k \in \{0, 1, 2, \ldots\}, \quad \sum_{i,k} \alpha_i^k < \infty \right\}.
\]

The set \( \mathcal{J} \) is countable, and, for every \( \alpha \in \mathcal{J} \), only finitely many of \( \alpha_i^k \) are not equal to zero. For \( \alpha \in \mathcal{J} \), define

\[
|\alpha| = \sum_{i,k} \alpha_i^k, \quad \alpha! = \prod_{i,k} \alpha_i^k!,
\]

and

\[
\xi_\alpha = \frac{1}{\sqrt{\alpha!}} \prod_{i,k} H_{\alpha_i^k}(\xi_{ik}),
\]

where

\[
H_n(t) = e^{t^2/2} \frac{d^n}{dt^n} e^{-t^2/2}
\]
is \( n \)-th Hermite polynomial. In particular, if \( \alpha \in \mathcal{J} \) is such that \( \alpha_i^k = 1 \) if \( i = j \) and \( k = l \), and \( \alpha_i^k = 0 \) otherwise, then \( \xi_\alpha = \xi_{jl} \).

**Definition 2.1.** The space \( L_2(\mathcal{F}^W_T) \) is called the Wiener Chaos space. The \( N \)-th Wiener Chaos is the linear subspace of \( L_2(\mathcal{F}^W_T) \), generated by \( \xi_\alpha, \quad |\alpha| = N \).

The following is a classical results of Cameron and Martin [2].

**Theorem 2.1.** The collection \( \{ \xi_\alpha, \alpha \in \mathcal{J} \} \) is an orthonormal basis in the space \( L_2(\mathcal{F}^W_T) \).

In addition to the original source [2], the proof of this theorem can be found in many other places, for example, in [6]. By Theorem 2.1 every element \( v \) of \( L_2(\mathcal{F}^W_T) \) can be written as

\[
v = \sum_{\alpha \in \mathcal{J}} v_\alpha \xi_\alpha,
\]

where \( v_\alpha = \mathbb{E}(v \xi_\alpha) \).
3. The Wiener Chaos Solution of the Passive Scalar Equation

Using the summation convention, define the operators \( A = 0.5(\nu \Delta + C^{ij}(0)D_iD_j) \) and \( M_k = \sigma_k^2 D_i \). Assume that \( \theta_0 \in L_2(\mathbb{R}^d) \) and define \( \theta_0(t, x) \) by

\[
\theta_0(t) = \theta_0 I(|\alpha| = 0) + \int_0^t A\theta_0(s, x) \, ds + \int_0^t \sum_{i,k} \sqrt{\alpha^k_i} M_k \theta_{\alpha^{-}(i,k)}(s) m_i(s) \, ds.
\]

Notice that for every function \( f \in H^1_2(\mathbb{R}^d) \),

\[
\sum_{k \geq 1} \|M_k f\|^2_{L^2(\mathbb{R}^d)} = (\sigma^2_{ij} D_i f, D_j f) = (C^{ij}(0) D_i f, D_j f),
\]

where \((\cdot, \cdot)\) is the inner product in \( L_2(\mathbb{R}^d) \). Since the matrix \((C^{ij}(0), i, j = 1, \ldots, d)\) is positive definite, we conclude that there exist positive numbers \( c_1, c_2 \) so that, for every function \( f \in H^1_2(\mathbb{R}^d) \),

\[
c_1 \|\nabla f\|^2_{L^2(\mathbb{R}^d)} \leq \sum_{k \geq 1} \|M_k f\|^2_{L^2(\mathbb{R}^d)} \leq c_2 \|\nabla f\|^2_{L^2(\mathbb{R}^d)}.
\]

**Theorem 3.2.**

1. For every \( \nu \geq 0 \) and every \( t \in [0, T] \), the series

\[
\sum_{\alpha \in J} \theta_\alpha(t, x) \xi_\alpha
\]

converges in \( L_2(\Omega; L^2(\mathbb{R}^d)) \) to a process \( \theta = \theta(t, x) \).

2. If \( \nu > 0 \), then, for every \( \phi \in C^\infty_0(\mathbb{R}^d) \), the process \( \theta(t, x) \) satisfies

\[
(\theta, \phi)(t) = (\theta_0, \phi) - 0.5\nu \int_0^t (\nabla\theta, \nabla\phi)(s) \, ds - 0.5 \int_0^t C^{ij}(0)(D_i\theta, D_j\phi)(s) \, ds
\]

\[
- \int_0^t (\sigma^2_{ij} D_i \theta, \phi) \, dw_k(s)
\]

with probability one for all \( t \in [0, T] \) at once, where \((\cdot, \cdot)\) is the inner product in \( L_2(\mathbb{R}^d) \). Also,

\[
\mathbb{E}\|\theta\|^2_{L^2(\mathbb{R}^d)}(t) + \nu \int_0^t \mathbb{E}\|\nabla\theta\|^2_{L^2(\mathbb{R}^d)}(s) \, ds = \|\theta_0\|^2_{L^2(\mathbb{R}^d)}.
\]

3. If \( \nu = 0 \), then, for every \( \phi \in C^\infty_0(\mathbb{R}^d) \), the process \( \theta(t, x) \) satisfies

\[
(\theta, \phi)(t) = (\theta_0, \phi) + 0.5 \int_0^t C^{ij}(0)(\theta, D_iD_j\phi)(s) \, ds + \int_0^t (\theta, \sigma^2_{ij} D_i\phi) \, dw_k(s)
\]

with probability one for all \( t \in [0, T] \) at once. Also,

\[
\mathbb{E}\|\theta\|^2_{L^2(\mathbb{R}^d)}(t) \leq \|\theta_0\|^2_{L^2(\mathbb{R}^d)}.
\]

**Remark 3.3.** Equalities (3.4) and (3.6) mean that \( \theta = \theta(t, x) \) is the solution of the transport equation in the traditional sense of theory of stochastic partial differential equations, that is, it is a strong solution in the stochastic sense, satisfying the corresponding equation in the generalized function sense. The solution is also unique in the class of \( L_2([0, T] \times \Omega; L^2(\mathbb{R}^d)) \) random functions, because any other solution will automatically have the same Wiener Chaos expansion. The uniqueness can, in fact, be established in a much wider class of generalized random functions.
The proof of Theorem 3.2 is based on the following lemmas.

**Lemma 3.4.** The system of equations (3.1) has a unique solution so that every \( \theta_\alpha \) is a smooth bounded function of \( x \) for \( t > 0 \) and, if \( T_1, t \geq 0 \), is the heat semigroup generated by the operator \( 0.5(\nu \Delta + C^3(0)D_iD_j) \), then, for every \( N \geq 0 \),

\[
\sum_{|\alpha|=N} |\theta_\alpha(t,x)|^2
\]

(3.8)

\[
= \sum_{k_1,\ldots,k_N=1}^{\infty} \int_0^t \int_0^{s_N} \ldots \int_0^{s_2} |T_{t-s_N}M_{k_N} \ldots T_{s_2-s_1}M_{k_1}T_{s_1} \theta_0(x)|^2 ds_1 \ldots ds_N.
\]

and

\[
\sum_{|\alpha|=N} |\nabla \theta_\alpha(t,x)|^2
\]

(3.9)

\[
= \sum_{k_1,\ldots,k_N=1}^{\infty} \int_0^t \int_0^{s_N} \ldots \int_0^{s_2} |\nabla T_{t-s_N}M_{k_N} \ldots T_{s_2-s_1}M_{k_1}T_{s_1} \theta_0(x)|^2 ds_1 \ldots ds_N.
\]

**Proof.** See Proposition A.1 in [9]. \( \square \)

**Lemma 3.5.** Assume that \( \nu \geq 0 \). Define \( \theta_N(t,x) = \sum_{n=0}^{N} \sum_{|\alpha|=n} \theta_\alpha(t,x) \xi_\alpha \). Then, for all \( t \in [0,T] \),

\[
\mathbb{E} ||\theta_N||^2_{L_2(\mathbb{R}^d)}(t) = ||\theta_0||^2_{L_2(\mathbb{R}^d)} - \nu \sum_{n=0}^{N} \sum_{|\alpha|=n} \int_0^t ||\nabla \theta_\alpha||^2_{L_2(\mathbb{R}^d)}(s) ds
\]

(3.10)

\[- \sum_{k_1,\ldots,k_{N+1}=1}^{\infty} \int_0^t \int_0^{s_N} \ldots \int_0^{s_2} ||M_{k_{N+1}}T_{s-N}M_{k_N} \ldots T_{s_2-s_1}M_{k_1} T_{s_1} \theta_0||^2_{L_2(\mathbb{R}^d)} ds_1 \ldots ds_N ds.
\]

**Proof.** By Lemma 3.4, after integration with respect to \( x \),

\[
\sum_{|\alpha|=N} ||\theta_\alpha||^2_{L_2(\mathbb{R}^d)}(t)
\]

(3.11)

\[
= \sum_{k_1,\ldots,k_N=1}^{\infty} \int_0^t \int_0^{s_N-1} \ldots \int_0^{s_2} ||M_{k_N}T_{t-s_N-1}M_{k_{N-1}} \ldots T_{s_2-s_1}M_{k_1} T_{s_1} \theta_0||^2_{L_2(\mathbb{R}^d)} ds_1 \ldots ds_N.
\]

If \( F_N(t) = \sum_{|\alpha|=N} ||\theta_\alpha(t)||^2_{L_2(\mathbb{R})} \), then

\[
\frac{d}{dt} F_N(t)
\]

(3.12)

\[
= \sum_{k_1,\ldots,k_N} \int_0^t \int_0^{s_N-1} \ldots \int_0^{s_2} ||M_{k_N}T_{t-s_N-1}M_{k_{N-1}} \ldots T_{s_2-s_1}M_{k_1} T_{s_1} \theta_0||^2_{L_2(\mathbb{R})} ds_1 \ldots ds_N - 1
\]

\[+ 2 \sum_{k_1,\ldots,k_N} \int_0^t \int_0^{s_N} \left( \langle AT_{t-s_N}M \ldots T_{s_1} \theta_0, T_{t-s_N}M \ldots T_{s_2-s_1}M_{k_N}T_{s_1} \theta_0 \rangle \right) ds_N.
\]
It remains to notice that, for every smooth function \( f = f(x) \),
\[
2(Af, f) = -\nu \|\nabla f\|_{L_2(\mathbb{R})}^2 - \sum_{k \geq 1} \|M_k f\|_{L_2(\mathbb{R})}^2.
\]
Equality (3.10) now follows. \( \square \)

Notice that (3.10) implies both the \( L_2(\Omega; L_2(\mathbb{R}^d)) \) convergence of the series
\[
\sum_{\alpha} \theta_{\alpha}(t, x) \xi_{\alpha}
\]
for every \( t \in [0, T] \) and inequality (3.7).

**Lemma 3.6.** If \( \nu > 0 \), then, for every \( t \in [0, T] \),
\[
\lim_{N \to \infty} \sum_{k_1, \ldots, k_{N+1}} \int_0^t \cdots \int_0^{s_2} \|M_{k_{N+1}} T_{s-s_N} M_{k_N} \cdots T_{s_2-s_1} M_{k_1} T_{s_1} \theta_0\|_{L_2(\mathbb{R}^d)}^2 ds_1 \cdots ds_N = 0.
\]

**Proof.** Define
\[
F_N(t) = \sum_{k_1, \ldots, k_{N+1}} \int_0^t \cdots \int_0^{s_2} \|M_{k_{N+1}} T_{s-s_N} M_{k_N} \cdots T_{s_2-s_1} M_{k_1} T_{s_1} \theta_0\|_{L_2(\mathbb{R}^d)}^2 ds_1 \cdots ds_N.
\]
By (3.2) and Lemma 3.4, \( F_N(t) \leq c_2 \sum_{|\alpha|=N} \int_0^t \|\nabla \theta_{\alpha}(s)\|_{L_2(\mathbb{R}^d)}^2 ds \). Lemma 3.5 then implies that the series \( \sum_{N \geq 0} F_N(t) \) converges for all \( t \in [0, T] \). Therefore, \( \lim_{N \to \infty} F_N(t) = 0 \) and the statement of the lemma follows. \( \square \)

Since (3.10) and (3.13) imply (3.5), to complete the proof of the theorem it remains
to establish (3.4) and (3.6). The necessary arguments are similar to the proof of
Theorem 3.5 in [10]

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