First, Rado introduced three results about empirical distribution

\[ \hat{F}_n(t) = \frac{1}{n} \sum_{i=1}^{n} 1_{(X_i \leq t)}. \]

1. **Strong Law of Large Numbers.** For all \( t \),

\[ \hat{F}_n(t) \xrightarrow{a.s.} F(t) \]

2. **Glivenko-Cantelli.**

\[ \| \hat{F}_n - F \|_\infty := \sup_{t \in \mathbb{R}} |\hat{F}_n(t) - F(t)| \xrightarrow{a.s.} 0. \]

3. **Dvoretzky-Kiefer-Wolfowitz.** For all \( \epsilon > 0 \),

\[ P \left( \sup_{i \in \mathbb{N}} |\hat{F}_n(i) - F(i)| > \epsilon \right) \leq 2e^{-2n\epsilon^2}. \]

Ideally, we want to replace \( \hat{F} \) with the empirical distribution of the eigenvalues and \( F \) with the semicircular distribution.

Next, we consider Haar measures. Before we define a Haar measure, we recall what a topological group is. A **topological group** \( G \) is a topological space and group such that the group operations of product

\[ G \times G \to G : (x, y) \mapsto xy \]

and taking inverses

\[ G \to G : x \mapsto x^{-1} \]

are continuous functions. Now, take \( (G, \cdot) \) be a locally compact, i.e. every point has a compact neighborhood, Hausdorff topological group. Define the left (resp. right) translate of \( G \) as \( gS = \{ g \cdot s : s \in S \} \) (resp. \( Sg = \{ s \cdot g : s \in S \} \)). A measure \( \mu \) is said to be left translation invariant if for all Borel subsets \( S \subseteq G \) and for all \( g \in G \), one has \( \mu(gS) = \mu(S) \). Right translation invariant measures are similarly defined.

Haar proved a theorem stating that there exists, up to a positive multiplicative constant, a unique countably additive, nontrivial measure \( \mu \) on the Borel subsets of \( G \) satisfying the following properties:

1. \( \mu \) is left translation invariant: \( \mu(gE) = \mu(E) \) for all \( g \in G \) and all Borel set \( E \).
2. \( \mu \) is finite on every compact set: \( \mu(K) < \infty \) for all compact \( K \).
3. \( \mu \) is outer regular on Borel sets \( E \): \( \mu(E) = \inf \{ \mu(U) : E \subseteq U, U \text{ open and Borel} \} \).
4. \( \mu \) is inner regular on open Borel sets \( E \): \( \mu(E) = \sup \{ \mu(K) : K \subseteq E, K \text{ compact} \} \).

Such a measure is called a **left Haar measure**. Using the general theory of Lebesgue integration, we define an integral for all Borel measurable functions \( f \) on \( G \), called the Haar integral:

\[ \int_G f(sx) \, d\mu(x) = \int_G f(x) \, d\mu(x). \]

Some examples of Haar measures (including the ones covered in class) are:

1. A Haar measure on \( (\mathbb{R}, +) \) which takes the value 1 on \([0, 1]\) is the restriction of lebesgue measure to the Borel subsets of \( \mathbb{R} \). This can be generalized to \( (\mathbb{R}^n, +) \).
2. If \( G = (\mathbb{R} \setminus \{ 0 \}, \times) \), then

\[ \mu(S) = \int_S \frac{1}{|t|} \, dt \]

for any Borel subset \( S \), is a Haar measure on \( G \).
3. If $G = GL(n, \mathbb{R})$, any left Haar measure is a right Haar measure and one such measure is given by

$$\mu(S) = \int_S \frac{1}{|\det(X)|^n} dX,$$

where $dX$ denotes the Lebesgue measure on $\mathbb{R}^{n^2}$.

4. On the unit circle $T$, consider the function $f : [0, 2\pi] \to T$ defined by $f(t) = (\cos(t), \sin(t))$. Then, $\mu$ defined by

$$\mu(S) \frac{1}{2\pi} m(f^{-1}(S)),$$

where $m$ is the Lebesgue measure, is a Haar measure.

We also discussed group actions in order to better understand the translation invariance in Haar integrals. Given a group $G$ and a set $X$, a (left) group action of $G$ on $X$ is a function $G \times X \to X$, $(g, x) \mapsto g.x$ satisfying the following two axioms:

1. $(gh).x = g.(h.x)$ for all $g, h \in G$ and all $x \in X$.
2. $1_G.x = x$ for all $x \in X$.

Another way to think of a group action is to consider it as a group homomorphism from $G$ into $\text{Sym}(X)$, the symmetric group of all bijections from $X$ to $X$. After a long debate, we did establish that

$$\int f(g) d\mu = \int f(a^{-1}g) d\mu.$$