Prof. Lototsky began the session with an exploratory question about the statement of the Semi-Circle Law: What kind of information do we get about the eigenvalues? The observation here is that the object in the theorem is not the same as the typical empirical measure, even though some numerical experiments suggest that the behaviors are similar.

Radoslav presented the following:

Given \( Z_{ij}, Y_i \) - two families of independent random variables, each family iid, with \( Z_{ij} \) mean zero and variance 1, along with \( \max(E|Z_{ij}|^k, E|Y_i|^k) < \infty \) \( \forall k \in \mathbb{N} \).

Construct Wigner matrix with \( Y_i \)s on main diagonal and \( Z_{ij} \)s off the main diagonal subject to symmetry, scaled by \( \frac{1}{\sqrt{N}} \):

\[
\begin{bmatrix}
  Y_1 & Z_{12} & Z_{13} & \cdots \\
  Z_{12} & Y_2 & Z_{23} & \cdots \\
  Z_{13} & Z_{23} & Y_3 & \cdots \\
  \vdots & \vdots & \vdots & \ddots \\
  \end{bmatrix}
\]
Order the eigenvalues $\lambda_i^N$: $\lambda_1^N \leq \lambda_2^N \leq \ldots \leq \lambda_N^N$.

Then $L_N := \frac{1}{N} \sum_i \mathcal{S}(\lambda_i^N) \xrightarrow{p} G(x) := \frac{1}{2\pi} \sqrt{4 - x^2} 1_{|x| \leq 2}$.

Lototsky's challenge question:

What can we say about the relationship, if any, between the $L_n$, as defined in the foregoing, and the empirical distribution $F_N := \frac{1}{N} \sum S_{x_i}$ for a series of actualizations $X_1, X_2, \ldots, X_N, \ldots$?

Lototsky: $L_N = \frac{1}{N} \sum S(\lambda_i^N) \xrightarrow{?} \lim_{M \to \infty} \frac{1}{M} \sum_{k=1}^M \frac{L_N^{(k)}}{M} \overset{?}{=} \text{Law}(L_N)$

Radoslav agreed to and will make/simulate histograms of eigenvalues considered in the foregoing, as well as $L_N$ for a large fixed $N$.

Haininy's presentation on Classical Matrix Groups:

**Skew Field:** Set $K$ with operation "+" and "·", having the property: $a \cdot (b + c) = (a \cdot b) + (a \cdot c) \forall a, b, c \in K$. $K$ is abelian group under +. Take 0 to denote the "additive" identity ($0 + a = a \forall a \in K$). Then $K - \{0\}$ is a group under the operation ·. Examples: $\mathbb{R}$ and $\mathbb{C}$ are fields, where as $\mathbb{R}$, $\mathbb{R}^2 \sim \mathbb{C}$, $\mathbb{R}^4 \sim \mathbb{H}$ are skew fields.
\[ f: \mathbb{C} \rightarrow \mathbb{R}^2 \text{ defined by } f(a + ib) = (a, b), \]
\[ g: \mathbb{H} \rightarrow \mathbb{C}^2 \text{ defined by } g(a + bi + cj + dk) = (a + bi), (c + di) \]

Note \( a + bi + cj + dk = (a + bi) + (c + di)j \) since \( ij = k \).

Let \( \mathbb{K} \) denote a skew field.

\[ \text{GL}_n(\mathbb{K}) := \{ A \in M_n(\mathbb{K}) : A \text{ is invertible} \} = \{ A \in M_n(\mathbb{K}) : \det(A) \neq 0 \} \]
(Recall: \( A \in M_n(\mathbb{K}) \) is invertible if \( \exists B \in M_n(\mathbb{K}) \) st \( AB = BA = I \)).

\[ f: \mathbb{C}^n \rightarrow \mathbb{R}^{2n}, \]
\[ <x, y>_{\mathbb{C}} = <f(x), f(y)>_{\mathbb{R}} + i<f(x), f(iy)>_{\mathbb{R}} \in \mathbb{C}, \]
\[ h := f \circ g: \mathbb{H}^n \rightarrow \mathbb{R}^{2n}, \]
\[ <x, y>_{\mathbb{H}} = <h(x), h(y)>_{\mathbb{R}} + i<h(x), h(iy)>_{\mathbb{R}} + \]
\[ + j<h(x), h(jy)>_{\mathbb{R}} + k<h(x), h(ky)>_{\mathbb{R}} \] \[ \in \mathbb{H}. \]

\[ \text{SL}_n(\mathbb{K}) := \{ A \in \text{GL}_n(\mathbb{K}) : \det(A) = 1 \} \]
\[ \text{O}_n(\mathbb{K}) := \{ A \in \text{GL}_n(\mathbb{K}) : <xA, yA> = <x, y> \ \forall x, y \in \mathbb{K}^n \}. \]

\( \text{O}(n) \) - orthogonal group when \( \mathbb{K} = \mathbb{R} \), \( \text{U}(n) \) - unitary group when \( \mathbb{K} = \mathbb{C} \),
\( \text{Sp}(n) \) - special group when \( \mathbb{K} = \mathbb{H} \).

Thm: \( \{ A \in \text{GL}_n(\mathbb{K}) : |\det(A)| = 1 \} = \text{O}_n(\mathbb{K}) \).