August 15, 2009

Mathematics of Physics and Engineering II: Homework answers

You are encouraged to disagree with everything that follows.

Homework 1.

PROBLEM 1.

(1) $\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = \langle -1,0,2\rangle - \langle 1,1,1\rangle = \langle -2,-1,1\rangle$, $\overrightarrow{PR} = \langle 0,-2,-2\rangle$, $\overrightarrow{PS} = \langle a,-1,-2a\rangle$.

(2) The vertex of the angle is P, so you need $\overrightarrow{PR}\cdot\overrightarrow{PS} = 0$, or $2+4a=0$. Therefore, $\boxed{a=-1/2}$.

(3) The area is $(1/2)|\overrightarrow{PQ}\times\overrightarrow{PR}|$ and

$$\overrightarrow{PQ}\times\overrightarrow{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & -1 & 1 \\ 0 & -2 & -2 \end{vmatrix} = \langle 4,-4,4\rangle = 4\langle 1,-1,1\rangle.$$

Consequently, the area is $\boxed{2\sqrt{3}}$.

(4) The normal vector to the plane is any vector parallel to $\overrightarrow{PQ}\times\overrightarrow{PR}$, for example, $\langle 1,-1,1\rangle$. Taking P as the point on the plane, we get the equation $(x-1)-(y-1)+(z-1)=0$ or $\boxed{x-y+z=1}$.

(5) You are welcome to compute the scale triple product using the determinant, but, given the work you already did, you do not have to compute another determinant. The answer is $4|1-a|$ (it has to be non-negative).

(6) You want the coordinates of S to satisfy the equation of the plane through P, Q, R, that is, $1+a-0+1-2a=1$ or $\boxed{a=1}$. You can also see it immediately from the volume formula.

(7) The direction vector for the line is the normal vector to the plane, that is, $\langle 1,-1,1\rangle$. Then the equation of the line is

$$\mathbf{r}(t) = \langle 1+t, -t, 1+t\rangle.$$

At the point of intersection, $(1+t)-(-t)+(1+t)=1$ or $t=-1/3$, so the point is $(2/3, 1/3, 2/3)$.

PROBLEM 2.

(1) $\mathbf{r}(1) = \langle 0,1,2\rangle$, so the coordinates of the point are $\boxed{(0,1,2)}$.

(2) $\mathbf{v}(t) = \dot{\mathbf{r}}(t) = \langle -2t, 3t^2, 2t\rangle$.

(3) $|\mathbf{v}(t)| = \sqrt{4t^2+9t^4+4t^2} = t\sqrt{8+9t^2}$.

(4) $\mathbf{a}(t) = \dot{\mathbf{v}}(t) = \langle -2, 6t, 2\rangle$.

(5) The particle is at $(0,1,2)$ when $1-t^2=0$ or $t=1$ (by assumption, $t\geq 0$). Therefore, the equation of the tangent line is $\mathbf{R}(u) = \langle 0,1,2\rangle + u\dot{\mathbf{r}}(1)$. Next, $\dot{\mathbf{r}}(1) = \langle -2,3,2\rangle$, and so the equation of the line is $\boxed{\mathbf{R}(u) = \langle -2u, 1+3u, 2-2u\rangle}$.

(6) You want the coordinates of the particle to satisfy the equation of the plane. Then $1+t^2-(1-t^2)=2$ or $t=1$ (remember, $t\geq 0$) So the point of intersection is $\boxed{(0,1,2)}$.

(7) According to the formula, the distance is

$$\int_0^1 |\mathbf{v}(t)|dt = \int_0^1 \sqrt{8+9t^2}\, t\, dt = \text{(simply guess antiderivative) } \frac{1}{27}(8+9t^2)^{3/2}\Big|_{t=0}^{t=1} = \boxed{\frac{17^{3/2}-8^{3/2}}{27}}.$$

Homework 2.

PROBLEM 1.

(1) $\nabla f(x,y) = \langle f_x(x,y), f_y(x,y)\rangle = \langle 4x-y-1, 2y-x+1\rangle$.

(2) The direction vector is $\mathbf{a} = \langle -1,-1\rangle$. Therefore, the rate is

$$\frac{\nabla f(1,1)\cdot\mathbf{a}}{|\mathbf{a}|} = \frac{\langle 2,2\rangle\cdot\langle -1,-1\rangle}{\sqrt{2}} = \boxed{-4/\sqrt{2}}.$$

The rate is negative, so the function is *decreasing* in that direction.

(3) The direction is given by $-\nabla f(1,1) = \langle -2,-2\rangle$, which is toward the origin. The rate of change is $-|\nabla f(1,1)| = -2\sqrt{2}$, which is, not surprisingly, the same as the rate of change toward the origin from the previous question.

(4) The path is $\mathbf{r}(t) = (x(t), y(t), z(t))$, where $\dot{x}(t) = 4x - y - 1$, $\dot{y}(t) = -x + 2y + 1$, $x(0) = y(0) = 0$, and $z(t) = 2x^2(t) - x(t)y(t) + y^2(t) - x(t) + y(t) - 1$.

On the topographic map (that is, the set of points $(x(t), y(t))$), the path is a parabola of the type $y = x^\alpha$, although twisted and turned. The reason is that the level sets of the function are ellipses, also twisted and turned.

Problem 2.

(1) 2π (the integrand is a potential field)
(2) 8 (Green's theorem)
(3) 5π (Stokes)

Problem 3.

(1) $\sqrt{2}(2\pi + (8\pi^3/3))$ (direct integration)
(2) 3π (Green's theorem is a better choice)
(3) $1/2$ (direct integration in spherical coordinates)
(4) 4π (a better approach is to close up the surface and use the divergence theorem, then subtract the extra flux through the top; the flux through the bottom is zero).

Problem 4. Draw the picture. For some orders of integration, you will have to break the region into several pieces.

Homework 3.

Problem 1.

$$\nabla^2 f = \frac{1}{r^2}\left(\frac{\partial}{\partial r} r^2 \frac{\partial f}{\partial r}\right) + \frac{1}{r^2 \sin^2\varphi}\frac{\partial^2 f}{\partial \theta^2} + \frac{1}{r^2 \sin\varphi}\frac{\partial}{\partial\varphi}\left(\sin\varphi \frac{\partial f}{\partial\varphi}\right).$$

Problem 2.

(1) $\frac{-11-2i}{25}$
(2) $\sqrt{2}\exp(-3i\pi/4 + 2i\pi n)$
(3) The four solutions are $z_1 = \sqrt[4]{2}\exp(i\pi/8)$, $z_2 = \sqrt[4]{2}\exp(i\pi/8 + i\pi)$ $z_3 = \sqrt[4]{2}\exp(-i\pi/8)$, $z_4 = \sqrt[4]{2}\exp(-i\pi/8 + i\pi)$
(4) $\int e^{-2x}\sin(3x)dx = Im\left(\int e^{(-2+3i)x}dx\right) = Im\left(\frac{e^{(-2+3i)x}}{-2+3i}\right) = (e^{-2x}/13)Im((\cos(3x) + i\sin(3x))(-2-3i)) = \boxed{(-e^{-2x}/13)(2\sin(3x) + 3\cos(3x))}$.
(5) $\sqrt[6]{-i} = \exp(-i\pi/12 + i\pi n/3$, $n = 0, 1, 2, 3, 4, 5$.
(6) One of them is $\sqrt{6}$.
(7) $\cos 5x = 16\cos^5 x - 20\cos^3 x + 5\cos x$.

Homework 4.

Problem 1.

(1) $z^3 - 2z + 1 = (x + iy)^3 - 2(x + iy) + 1 = x^3 + 3ix^2y - 3xy^2 - iy^3 - 2x + 2iy + 1$, so $\boxed{Re(f) = x^3 - 3xy^2 - 2x + 1,\ Im(f) = 3x^2y - y^3 + 2}$. It is analytic everywhere, because it is a polynomial (or you can verify the Cauchy-Riemann equations).
(2) $f(z) = (x + iy)(\cos y + i\sin y)e^x$; $Re(f) = e^x(x\cos y - y\sin y)$. It is analytic.
(3) $f(z) = (e^{iz} + e^{-iz})/2$; $Re(f) = \cos x \cosh y$ (hyperbolic functions appear here).
(4) If f is analytic everywhere and $Re(f) = 0$, then Cauchy-Riemann equations imply that $v = Im(f)$ satisfies $v_x = v_y = 0$ everywhere, or $\boxed{v = Im(f) = const}$.

(5) If $u(x, y) = ax^3 + bxy$, then $u_{xx} + u_{yy} = 6ax$. Therefore, $a = 0$ and b can be any real number $u(x, y) = bxy$. To find conjugate harmonic v we write $u_x = by = v_y$, $u_y = bx = -v_x$; one of the solutions is $\boxed{v(x, y) = b(y^2 - x^2)/2}$. Note that the resulting $f(z) = u(x, y) + iv(x, y)$ is $f(z) = -ibz^2/2$.

Problem 2. Under the map $f(z) = 1/z$:

(1) $\{z : |z| < 1\}$ becomes $\{z : |z| > 1\}$ (kind of obvious)

(2) $\{z : Re(z) > 1\}$ becomes $\{z : |z - 1/2| < 1/2$: the circle $(x - 1/2)^2 + y^2 = 1/4$ is the image of the line $x = 1$.
(3) $\{z : 0 < Im(z) < 1\}$ becomes $\{z : Im(z) < 0$ and $|z + 1/2i| > 1/2\}$: again, the circle $x^2 + (y + 1/2)^2 = 1/4$ is the image of the line $y = 1$.

Homework 5.

PROBLEM 1.

(1) $R = (13/10)^{1/4}$
(2) $R = 37^{-1/4}$
(3) $R = 2^{2/3}$
(4) $R = \sqrt{e}$
(5) $R = 3e^2\sqrt{3}/4$.

PROBLEM 2.

(1) $\sum_{n\geq 0} \frac{(-2)^n}{3^{n+1}} z^n, R = 3/2$
(2) $1 + (z - 1) + \sum_{n\geq 2} \frac{(-1)^n}{2^n}(z - 1)^n,\ R = 2$
(3) $\sum_{n\geq 0}(n + 1)2^n z^n,\ R = 1/2$ (differentiate a suitable function)
(4) $f(z) = \dfrac{1}{2i}\left(\dfrac{1}{z+1-i} - \dfrac{1}{z+1+i}\right)$ (partial fractions). You can take it from here. $R = \sqrt{2}$.

PROBLEM 3.

(1) $\dfrac{1}{2z} + \displaystyle\sum_{n\geq 0} \dfrac{z^n}{2^{n+2}}$
(2) $\displaystyle\sum_{n\geq 1} \dfrac{(-1)^n 2^{n-1}}{(z-2)^{n+1}}$
(3) $-\dfrac{1}{2(z-2)} + \displaystyle\sum_{n\geq 0}(-1)^n \dfrac{(z-2)^n}{2^{n+2}}$
(4) $\dfrac{1}{2}\displaystyle\sum_{n\geq 0} \dfrac{1}{(z+1)^{n+1}} + \dfrac{1}{6}\sum_{n\geq 0} \dfrac{(z+1)^n}{3^n}$

PROBLEM 4.

(1) removable
(2) removable
(3) second-order pole
(4) essential
(5) not an isolated singularity

Homework 6.

PROBLEM 1.

(1) 3
(2) 33
(3) $-\pi$
(4) -3π
(5) $-1/6$.

PROBLEM 2.

(1) $8\pi^2$
(2) $2\pi/3$
(3) $\pi\sqrt{3}/72$

PROBLEM 3.

(1) $w(z) = \sum_{k\geq 1} a_k z^{3k}$, where $a_k = \prod_{m=1}^{k}(3m(3m - 1)), k \geq 1$.
(2) $w(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(k!)^2} \left(\frac{z}{2}\right)^{2n}$.

(3) $w(z) = z^2 - 1$.

Problem 4.

(1) $\lambda = 2n$ (Hermite polynomials)
(2) $\lambda = n$ (Chebyshev polynomials of the first kind)
(3) $\lambda = n(n+1)$ (Legendre polynomials)

The main thing to keep in mind is that if $w(z) = \sum_{k\geq 0} a_k z^k$, then

$$w''(z) = \sum_{k\geq 2} k(k-1)a_k z^{k-2} = \sum_{k\geq 0}(k+1)(k+2)a_{k+2}z^k.$$

Homework 7.

Problem 1. $\sum_{n\geq 1} z^n/n$. On the boundary you have $|z - z_0| = R$, so absolute convergence of $\sum_n a_n(z-z_0)^n$ even at one point of the boundary implies convergence of $\sum_n |a_n|R^n$.

Problem 2.

(1) 0, not uniform: $(1-(1/n))^n \to 1/e \neq 0$;
(2) 0, uniform: $|\sin(x/n)| \leq |x/n| \leq 4/n$;
(3) 1, not uniform: if $x = 1/n$, you get $1/2$;
(4) x^2, uniform: $|nx^2/(n+x) - x^2| \leq 1/n$.

Problem 3.

(1) absolutely but not uniformly
(2) absolutely and uniformly
(3) absolutely and uniformly
(4) absolutely but not uniformly

Problem 4. Draw the pictures. Then everything is clear:

(1) $g(x) = (2/\pi)f(\pi(x+1/2)) - 1$,
(2) $g(x) = 2f(2\pi x) - 1$,
(3) $g(x) = (1/2\pi)\Big(f(2\pi x - \pi) + \pi\Big)$.

Problem 5.

(1) $f(x) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{k\geq 0}\frac{\cos((2k+1)x)}{(2k+1)^2}$, $g(x) = \frac{8}{\pi^2}\sum_{k\geq 0}\frac{(-1)^k\sin(\pi(2k+1)x)}{(2k+1)^2}$.

(2) $f(x) \sim \frac{1}{2} + \frac{2}{\pi}\sum_{k\geq 0}\frac{\sin((2k+1)x)}{2k+1}$, $g(x) \sim \frac{4}{\pi}\sum_{k\geq 0}\frac{\sin(2\pi(2k+1)x)}{2k+1}$.

(3) $f(x) \sim 2\sum_{k\geq 1}\frac{(-1)^{n+1}}{n}\sin(nx)$, $g(x) \sim \frac{1}{2} - \frac{1}{\pi}\sum_{k\geq 1}\frac{1}{n}\sin(nx)$.

(A discontinuous function is not equal to its Fourier series. This is why sometimes I write = and sometimes $\sim$.)

Problem 6. Only option (b) results in the *continuous* periodic function. For continuous periodic functions, the Fourier series converges better than for discontinuous; therefore, I would go with option (b).

Problem 7.

(1) $\pi/4$
(2) $\pi^4/96$
(3) $\pi^2/8$
(4) $\pi^2/6$

Problem 8. The graph is the periodic (with period 2) extension of f, except that $S_f(k) = 0$ for $k = 0, \pm 1, \pm 2, \ldots$. Therefore, $S_f(3) = 0$ and $S_f(5/2) = S_(1/2) = 1$.

Homework 8.

Problem 1.

(1) $\hat{f}(w) = \frac{1}{\sqrt{2\pi}(-2+iw)}$ (direct integration).

(2) $\sqrt{2\pi}\hat{f}(w) = \frac{i}{w}(be^{-iwb} - ae^{-iaw}) + \frac{1}{w^2}(e^{-iwb} - e^{-iaw})$ (direct integration by parts).

(3) $\hat{f}(w) = \sqrt{2/\pi}\frac{\sin w}{w(1+w^2)}$.

Problem 2. $\hat{f}(w) = \sqrt{2/\pi}\frac{1}{1+w^2}$. Then, using the formula for the inverse Fourier transform, and keeping in mind that $e^{iwx} = \cos(wx) + i\sin(wx)$, where cos is even and sin, odd, we get $\int_0^\infty \frac{\cos(wx)}{1+w^2}dw = \frac{\pi}{2}e^{-|x|}$. Also, the result means that the Fourier transform of $g(x) = 1/(1+x^2)$ is $\hat{g}(w) = \sqrt{\pi/2}e^{-|w|}$. Therefore, by Parseval's identity,

$$\int_{-\infty}^{+\infty} \frac{dx}{(1+x^2)^2} = \frac{\pi}{2}\int_{-\infty}^{+\infty} e^{-2|w|}dw = \pi\int_0^{+\infty} e^{-2w}dw = \frac{\pi}{2}.$$

Problem 3. $g(x) = f(\sqrt{2}x)$, so $\hat{g}(w) = \frac{1}{\sqrt{2}}\hat{f}(w/\sqrt{2}) = (\sqrt{2})^{-1}e^{-x^2/4}$.

Problem 4. $Y(z) = \frac{zX(z)}{z-a}$.

Homework 9.

Problem 1. $F(x + \cos t)$, where F is an arbitrary continuously differentiable function.

Problem 2. $F(xy)+G(x/y)$, where F, G are arbitrary twice continuously differentiable functions.

Problem 3.

(1) After separation of variable, you find that $u(x,t) = \sum_{n\ge1} A_n e^{-a_n^2 t}\sin(a_n x)$, and the second boundary condition implies $a_n = (n-0.5)\pi$.
(2) After separation of variable, you find that $u(x,t) = \sum_{n\ge1} A_n e^{-a_n^2 t}\sin(a_n x)$, and the second boundary condition implies $a_n = \tan(a_n)$.
(3) Note that $u_0(x,t) = 1-x$ is a solution of the equation, and it satisfies the boundary conditions. Since $u_0|_{t=0} = 1-x$, we find that $u(x,t) = u_0(x,t)+v(x,t)$, where v is the solution of $v_t = v_{xx}$, $t > 0$, $0 < x < 1$, $v|_{t=0} = x - 1 + u(0,x)$, $v|_{x=0} = 0$, $v|_{x=1} = 0$, which you know how to solve.
(4) Note that $v(t,x) = u(t,x)\exp(-\int_0^t s^2ds) = u(t,x)e^{-t^3/3}$ satisfies $v_t = v_{xx}$ with the same initial and boundary conditions.

Problem 4.

(1) Use the formula for the solution and the results about the Fourier transform to get

$$u(x,t) = \frac{1}{\sqrt{2\pi(1+t)}}e^{-\frac{x^2}{2(1+t)}}.$$

(2) This follows from the formula for the solution because the heat kernel is positive and integrates to one.

Homework 10.

Problem 1.

(1) $F''/F + G''/G = 0$, $F'' = cF$, $G'' = -cG$. If $c = a^2$, we get $F(x) = C_1\sinh(ax)+C_2\cosh(ax)$, $G(y) = C_3\sin(ay) + C_4\cos(ay)$ and $u(x,y) = (C_1\sinh(ax) + C_2\cosh(ax))(C_3\sin(ay) + C_4\cos(ay))$, C_i are arbitrary constants. If $c = -a^2$, then just switch x and y in the above expression for u. If $c = 0$, then $u = (C_1 + C_2x)(C_3 + C_4y)$.
(2) $F'/(x^2F) = G'/(y^2G) = 3c$, $u(x,y) = Ae^{c(x^3+y^3)}$.
(3) $(F'/F) - x = -(G'/G) + y = c$, $u = Ae^{c(x-y)+(x^2+y^2)/2}$.
(4) $xF'/F = -2yG'/G = c$, $u = Ax^ce^{-y^2/c}$, $c \ne 0$; $u = 0$ is also a solution; it is included in the family.
(5) $u = x/t$

Problem 2.

(1)

$$u(x,t) = (1/4) + \frac{1}{4\pi}\sin(4\pi t)\cos(2\pi x) - \sum_{n\geq 0}\frac{2\cos(2(2n+1)\pi x)}{\pi^2(2n+1)^2}\cos(4(2n+1)\pi t).$$

(2) $u(x,t) = \cos(2t)\sin(2x) + \sin(t)\sin(x) + \sin(x)$.

(3) No solution exists: the initial speed $u_t(x,0)$ cannot be written in the required form $\sum_{k\geq 1} 2\pi k b_k \cos(\pi k x)$ because $\int_0^1 f(x)dx \neq 0$.

PROBLEM 3. The answer is 0.

Homework 11.

PROBLEM 1.

(1) $u(x,y) = \frac{\sin(x)\sinh(y/2)}{\sinh(\pi/2)}$.

(2) $u(x,y) = \frac{4}{\pi^2}\sum_{m,n=1}^{\infty}\frac{(1-(-1)^n)(1-(-1)^m)}{(m^2+n^2)mn}\sin(mx)\sin(ny)$

(3) $u(r) = \sum_{n\geq 1} c_n J_0(\alpha_n r)$, $r = \sqrt{x^2+y^2}$,

$$c_n = \frac{\int_0^1 J_0(\alpha_n r) r dr}{\alpha_n^2 \int_0^1 J_0^2(\alpha_n r) r dr},$$

where $0 < \alpha_1 < \alpha_2 < \alpha_3 < \dots$ are the zeros of Bessel's function J_0.

(4) No solution exists.

PROBLEM 2.

(1) With a radially-symmetric initial condition, the solution should be radially symmetric as well (no dependence on θ). Therefore the basis functions come from J_0. Writing $\lambda_k = -\alpha_k^2/4$, $\varphi_k(r) = J_0(\alpha_k r/2)$, where $0 < \alpha_1 < \alpha_2 < \alpha_3 < \dots$ are the zeros of Bessel's function J_0, we get

$$u(t,r,\theta) = \sum_{k\geq 1} f_k e^{\lambda_k t}\varphi_k(r),$$

where

$$c_k = \frac{\int_0^2 \varphi_k(r) f(r) r dr}{\int_0^2 \varphi_k^2(r) r dr}$$

(2) This time, the initial condition suggests that the basis functions come from J_1 (recall that the general basis function is $J_N(\cdot)\psi(N\theta)$, where ψ is either sin or cos). Writing $\varphi_k(r) = J_1(\beta_k r)$, where $0 < \beta_1 < \beta_2 < \beta_3 < \dots$ are the zeros of Bessel's function J_1, we get

$$u(t,r,\theta) = \frac{\cos(\theta)}{\pi}\sum_{k\geq 1} f_k \cos(\beta_k t)\varphi_k(r),$$

where

$$c_k = \frac{\int_0^1 \varphi_k(r) f(r) r dr}{\int_0^1 \varphi_k^2(r) r dr}.$$

The factor $1/\pi$ comes from $\int_0^{2\pi}\cos^2\theta\, d\theta = \pi$.

PROBLEM 3. Just do it.

PROBLEM 4. For example, Bessel functions, Cauchy problem in PDEs and two theorems in complex analysis, Dirichlet problem in PDEs, Euler formula in complex analysis, Fourier series and transform, Gauss integers $n + im$, Gibbs phenomenon, Green' theorem, Laplace transform and equation, Poisson equation, Riemann surface, Stokes's theorem.